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A New Hypothesis Concerning Children's Fractional Knowledge

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Abstract: The basic hypothesis of the teaching experiment, The Child's Construction of the Rational Numbers of Arithmetic (Steffe & Olive, 1990) was that children's fractional schemes can emerge as accommodations in their numerical counting schemes. This hypothesis is referred to as the reorganization hypothesis because when a new scheme is established by using another scheme in a novel way, the new scheme can be regarded as a reorganization of the prior scheme. In that case where children's fractional schemes do emerge as accommodations in their numerical counting schemes, I regard the fractional schemes as superseding their earlier numerical counting schemes. If one scheme supersedes another, that does not mean the earlier scheme is replaced by the superseding scheme. Rather, it means that the superseding scheme solves the problems the earlier scheme solved but solves them better, and it solves new problems the earlier scheme didn't solve. It is in this sense that we hypothesized children's fractional schemes can supersede their numerical counting schemes and it is the sense in which we regarded numerical schemes as constructive mechanisms in the production of fractional schemes (Kieren, 1980).

Keywords: Equi-partitioning scheme; Numerical counting scheme; Connected numbers

The concept of scheme

The reorganization hypothesis is untenable if counting is regarded only as activity. Focusing only on the activity of counting, however, does not provide a full account. von Glasersfeld (1980), in a reformulation of Piaget's concept of scheme, elaborated the concept of scheme in a way that opens the possibility of focusing on what may go on prior to observable action. It also opens the possibility that the action of a scheme is not sensory-motor action, but interiorized action that is executed with only the most minimal sensory-motor indication. Finally, it opens the possibility to focus on the results of the scheme's action and how those results might close the child's use of the scheme.

The parts of a scheme

According to von Glasersfeld, a scheme consists of three parts. First, there is an experiential situation; an activating situation as perceived or conceived by the child, with which an activity has been associated. Second, there is the child's specific activity or procedure associated with the situation. Third, there is a result of the activity produced by the child. The situation of a scheme is an experiential situation as perceived or conceived by the child rather than by the observer. Piaget (1964), in a critique of the classical S → R schema, commented that the observer's stimulus is a stimulus for a child:

> only to the extent that it is significant, and it becomes significant only to the extent that there is a structure which permits its assimilation, a structure which can integrate this stimulus but which at the same time sets off the response. (p. 18)

So, the first part of a scheme is established by assimilation. But I do not assume that a structure "exists" somewhere in the mind in its totality and as an object. Rather, I assume that operations used in past activity are used to assemble a recognition template which is used in creating an experiential situation that may have been experienced before. When it is clear from context, I refer to the recognition template as an assimilating structure.

The numerical counting schemes

In other sources (Steffe & Cobb, 1988; Steffe, 1992, 1994) I have explained the construction of four distinctly different numerical counting schemes that I called the initial number sequence, the tacitly nested number sequence, the explicitly nested number sequence, and the generalized number sequence. As I am concerned with only the third of these four numerical counting schemes in this paper, I restrict my comments to that scheme. I referred to the numerical counting schemes as number sequences to emphasize the first part of the schemes. My concept of a number sequence is a sequence of abstract unit items that contain records of counting acts. Upon being produced by the operations used in its construction, an abstract unit item is much like a slot that has been lifted from the results of unitizing an image of a perceptual unit item (cf. von Glasersfeld, 1981 for a model of the operation of unitizing). I do not regard abstract unit items as static objects stored somewhere in the mind that can be used as material in further operating. Rather, I think of them as being produced upon activation of the operations that produced them. An activated number sequence is somewhat like a resonating tuning fork with the stipulation that its resonating creates an anticipation of counting. In that case where the child produces an image of counting, the image might be simply some minimal re-presentation of the involved number words that symbolize counting, or it might be a plurality of flecks that symbolize countable perceptual items. The hypothetical *slots* that contain records of experience that are used in producing operative images are realized only upon activation of the program of operations that produce them, and they are realized only as images produced by means of the records contained in them. The child who has constructed the explicitly nested number sequence can take these images as material of using the number sequence. That is, the number sequence can be taken as material of its own operating. This permits the child to produce two number sequences, one to operate on and one to operate with.

In a situation, say, where a child has put a handful of pennies with 19 pennies, and then counts all of the pennies and finds that there are 27, the child might become aware of the unknown numerosity of the added pennies and thereby form a goal to find how many pennies were added. This goal may lead the child to reconstitute the situation using its number sequence. In the reconstitution, the child might regenerate the experience of counting to twenty-seven by forming an image of the results of counting, which may be nothing more than an awareness of the number words uttered to "twenty-seven." Because the units of twenty-seven have been constituted as iterable units, the image of twenty-seven may be of a singular unit item that can be iterated twenty-seven times. This provides the child with great economy in thinking, because to regenerate an experience of counting twenty-seven items, the child doesn't need to regenerate a plurality of counting acts.

"Twenty-seven," for example, may stand in for a composite unit item (or slot) that could be filled with twenty-seven unit items by counting to "twenty-seven." The child knows that it can count to "twenty-seven" and doesn't need to carry out the activity. This iteration of a unit item by counting is indicated when the child counts its acts of counting from "nineteen" up to and including "twenty-seven" starting with "one." Of course, taking the remainder of nineteen in twenty-seven as a given means that the child disembeds this part of twenty-seven from twenty-seven and unites the image of the sequence of number words "twenty, twenty-one, ... , twenty-seven" into a composite unit.

These are the principal operations of the scheme that I refer to as the explicitly nested number sequence. I want to first highlight the ability of the child to form an image of twenty-seven as a singular unit item that can be iterated by counting twenty-seven times to fill out a composite unit containing twenty-seven items. The child can "go" to the implication of its operative image and generate an image of the composite unit that is produced by the activity of counting. So, I don't regard the composite unit structure as an object the child has 'stored' somewhere in memory. Rather, I regard it as a product of an ensemble of possible operations that is symbolized by the word "twenty-seven." In regarding it in this way, I am in agreement with Dörfler (1996) about not pursuing a theory of mind where the mind has or contains mental objects corresponding to numbers, natural or rational. Second, I want to highlight the ability of the child to disembed an image of the continuation of nineteen in twenty-seven and to unite the items of the image into a composite unit without destroying the composite unit structure that contains it. Iterating a unit item and disembedding a numerical part from a numerical whole are two principal operations of the numerical counting scheme that I refer to as the explicitly nested number sequence. A third is the operation of taking the number sequence as its own material of operating. This recursive operation emerges at the stage of the explicitly nested number sequence because it was precisely this operation that transformed the preceding number sequence into the explicitly nested number sequence.

The equi-partitioning scheme

I emphasize that when I use the phrase "the explicitly nested number sequence," I am referring to the first part of a numerical counting scheme. In this case, the activity of counting is interiorized activity and it is no exaggeration to say that it is contained in the first part of the counting scheme. That is, the explicitly nested number sequence is an interiorized counting scheme. This number sequence is an example of what Piaget (1964) meant when he commented that "there is a structure which integrates the stimulus but which at the same time sets off the response" when speaking of a stimulus from the point of view of the child. The first problem which had to be solved in establishing the reorganization hypothesis was to explain how children might come to use their number sequences, not to produce a composite unit of counted items, but to use this composite unit as a template for partitioning a continuous unit item. Toward that end, I start with a discussion of two protocols extracted from two teaching episodes held on the 28th of April and the 1st of May with Jason and Patricia during their third grade in school for the purpose of illustrating how these two children used their number concepts in constructing what I refer to as the equi-partitioning scheme. Both children were judged to have already constructed the explicitly nested number sequence at the beginning of the teaching experiment in October of their third grade in school. The children were taught together during their third grade using two computer tools we designed for the experiment: TIMA (toys and sticks)[1] (Biddlecomb, 1994; Olive, 2000; Olive & Steffe, 1994).

We did not decide a priori how to bring forth the children's use of their number sequences in their construction of partitioning schemes. Rather, the two children independently used their number sequences in partitioning activity, and it was not suggested by the teacher-researcher in the teaching episodes. From our point of view, it occurred fortuitously in that the situation of learning was not designed for that purpose.

Breaking a stick into two equal parts

Using TIMA (sticks), the children were asked to break a stick into two substicks of equal length. After drawing a stick, the children were asked to cut it into two equal parts. Jason cut the stick into two pieces as shown in Fig. 1. The two pieces were obviously of different lengths and the two children made a visual comparison between them. The teacher asked, "How do you know that the two pieces are of the same size?" There were several suggestions for how to test to find if the two pieces were of the same size, such as Jason's comment to "copy the biggest one and then copy them again." He then said, "no," shaking his head. After Jason' suggestion, the two children sat in silent concentration. The children had formed a goal of finding a way to test whether the two pieces were of equal size, but they seemed to have no action they could use to reach their goal. So, the teacher suggested to Patricia that she draw a shorter stick that would be easier to divide visually. After Patricia drew this stick, the actions the children contributed that are reported in Protocol I were not suggested by the teacher. Rather, they arose independently from the children. In the protocol, "*T*" stands for "teacher," "*P*" for "Patricia," and "*J*" for "Jason."

Figure 1. Cutting a stick into two equal parts using visual estimation.

Protocol I: Using a number sequence to break a stick into two equal pieces.

T: (After Patricia had drawn a stick about one decimeter in length) Now, I want you to break that stick up into two pieces of the same size.

P: (Places her right index finger on the right endpoint of the stick, then places her right middle finger to the immediate left of her index finger. Continues on in this way, walking her two fingers along the stick in synchrony with uttering) one, two, three, four, five. (Stops when she is about one-half of the way across the stick.)

J: (Places his right index finger where Patricia left off; uses his right thumb rather than his middle finger to begin walking along the stick. Changes to his left index finger rather than his right thumb after placing his thumb down once. Continues on in this way until he reaches the left endpoint of the stick) six, seven, eight, nine, ten. (Then) there's five and five (Smiles with satisfaction).

P: (Smiles also.)

Patricia independently introduced the action of walking her fingers along the stick until she arrived at a place she regarded as one-half of the way. Jason picked up counting where

[1] These computer tools (TIMA, tools for interactive mathematical activity) were programmed by Dr. Barry Biddlecomb in consultation with Dr. John Olive and myself.

Patricia left off, which solidly indicates that he assembled meaning for Patricia's method of establishing equal pieces of the stick. Patricia's counting activity seemed meaningful to him and he could be said to engage in consensual mathematical activity with Patricia. In watching the video tape, it seemed that she stopped counting at "five" because she reached a place that she regarded as one-half of the way across the stick. Patricia, as well as Jason, now had a way to at least justify where the stick should be cut so that the two pieces would be of the same size. The pleased look on their faces indicated that they had achieved their goal.

Testing the viability of a new concept of partitioning

Explaining how Patricia established a blank stick as a situation of counting involves her projecting units into the stick in such a way that she imagined the stick broken into two equal sized pieces, where each piece was in turn broken into an indefinite numerosity of pieces of the same size. The fact that Patricia counted indicates that she was aware of an unknown numerosity of pieces prior to counting. So, at least in the case of the two children, a very basic condition for the reorganization hypothesis to be viable had been established. But it only established the hypothesis as a possibility. The children used their composite units as templates for partitioning a stick into equal and connected parts and I regarded these templates as assimilating structures of a possible fractional scheme. But the children had no reliable activity they could use to partition a stick into any definite numerosity of pieces. Based on the insight that the children independently used their number sequence in partitioning, I *hypothesized that the operations of iterating and partitioning are parts of the same psychological structure*. The hypothesis would be confirmed if any single part of a partitioning could be used to reconstitute the unpartitioned whole by iterating the part.

Based on the concept of partitioning as a psychological structure that included both operations of breaking a continuous unit into equal sized parts and iterating any of the parts to reconstitute the whole, a situation of learning was designed in a test of the hypothesis. If my concept of partitioning was viable, then it should have been possible to engender an accommodation in the children's number sequences in such a way that they would use the parts of the stick they produced in partitioning in judging equal parts rather than their fingers.

Protocol II: Breaking off one of four equal parts of a stick.

T: Let's say that the three of us are together and then there is Dr. Olive over there. Dr. Olive wants a piece of this candy (the stick), but we want to have fair shares. We want him to have a share just like our shares and we want all of our shares to be fair. I wonder if you could cut a piece of candy off from here (the stick) for Dr. Olive.

J: (Using MARKS, makes three marks on the stick, visually estimating the place for the marks.)

P: How do you know they are even? There is a big piece right there.

J: I don't know. (Clears all marks and then makes a mark indicating one share. Before he can continue making marks, the teacher-researcher intervenes.)

T: Can you break that somehow (the teacher-researcher asks this question to open the possibility of iterating)?

J: (Using BREAK, breaks the stick at the mark; then makes three copies of the piece; aligns the copies end-to-end under the remaining piece of the stick starting from the left endpoint of the remaining piece as in Fig. 2.)

T: Why don't you make another copy (this suggestion was made to explore if Jason regarded the piece as belonging to the three copies as well as to the original stick)?

J: (Makes another copy and then aligns it with the remaining part of the original stick.

He now has the four copies aligned directly beneath the original stick which itself is cut once. The four pieces joined together were slightly longer than the original stick as in Fig. 3.)

Figure 2. Jason testing if one piece is one of four equal pieces.

Figure 3. Jason's completed test.

Jason copied the part he broke off from the stick three times in a test to find if three copies would reconstitute the remaining part. This was crucial because it was the basis of my inference that he anticipated producing the three copies prior to their production. To do this, he would need to repeatedly use the operations involved in making a stick in visualizing a stick, a use that is essential in iterating the stick. This repeated use of his operation of making a stick in visualizing a stick was presaged by his comment "copy the biggest one and copy them again" preceding Protocol I.

Comparing the three copies with the remaining part of the original stick indicates that Jason took the three copies as a term of comparison; that is, as a unit containing three units. This opens the possibility that he could unite a current copy of the part with those he had previously made. The possibility is confirmed by his making another copy and then aligning it with the remaining part of the original stick after the teacher suggested that he make another copy. Jason's way of operating in Protocol II was a modification of the operations that constituted his concept of four.

Patricia, in the same teaching episode, demonstrated that she too could operate in the way Jason operated in Protocol II. I call the scheme that Jason and Patricia constructed an *equi-partitioning scheme*. It is crucial to understand that the independently contributed language and actions of the children served in my construction of this scheme. It is crucial because I regard schemes as goal-directed systems of action or operation.

Connected numbers[2]

The *result* of Jason's mathematical activity in Protocol II can be regarded as a *connected number*, four. I turn now to a further analysis of how children might construct connected

[2] I would like to thank Dr. Ron Tzur for making his transcriptions of the video-tapes which serve as a basis for Protocols III through XI available for my use in viewing the video-tapes.

numbers. Due to scheduling difficulties, Patricia was paired with another child, Ricardo, during their fourth grade and Jason was paired with the child, Laura, during their fourth and fifth grades. So, I analyze the connected numbers that Laura constructed for comparison and contrast with those that Jason constructed.

Making connected numbers using segmenting

In Protocol II, I assume that Jason's estimate was embedded in his use of his composite unit, four, to mentally project four separated but connected units into the stick. This seemed to occur as one composite act rather than as four individual and sequential acts and this is why I regarded it as an act of partitioning. In the first teaching episode of the year held with Jason and Laura on the 12th of October, there is indication of breaking a stick into three parts by sequential acts.

Protocol III: Sharing one stick equally among three people.

T: (Asks the children to share a stick among three people.)

J: (Marks a stick into three parts that were obviously not the same length.)

T: (To Laura) do you think they are equal?

L: (Shakes her head "no." She then erases the right most mark and reactivates Marks. She then runs the cursor from the left endpoint to the mark in a uniform motion and continues on until reaching a place she thinks marks off a part equal to the first part and makes a mark. She then continues on running the cursor with the same uniform motion to the rightmost endpoint.)

J: (Right after Laura has finished sweeping) That one is bigger.

L: (Erases the second mark) This is going to take a long time! (Measures more slowly with her sweeping motion over the first part of the stick and continues along the stick until she makes another mark.)

J: (Says he wants to do something, but the teacher intervenes.)

T: How could you check that now? (Encourages Laura to use Image.)

L: (Pulls out an image of the first part and places it below the first part. She then pulls out a second image of the first part and places it just beyond the first image with end points almost touching. She then pulls another image out from the first part and tries to place it just beyond the second image, but she releases the mouse unintentionally so the third image is to the right and below the two others. She tries to drag the images, but they are not moveable. So, with the teacher's directions, she uses the menu item Clean Up to remove the images.)

Laura's act of partitioning the stick into what she considered as three equal parts was quite different than Jason's partitioning a stick into four parts. Laura used one three times in segmenting the stick rather than three units once to partition the stick. Even more interesting is the realization that Laura used Jason's initial estimate in gauging the length of the two other parts of the stick she made. In fact, her way of gauging the length of the two other parts indicates that her length concept arose in situations other than her use of her number concepts to partition unmarked sticks. That children construct quantitative operations in the case of continuous quantity as well as discrete quantity has been adequately demonstrated by Piaget, Inhelder, and Szeminska (1960), and it is fundamental to the reorganization hypothesis because operations implied by the number sequence are specialized operations and cannot possibly be used to fully explain quantitative concepts like length.

Based on Laura's actions in Protocol III, it is possible to give an account of her concept of length. Running the cursor over the first part of the stick indicates that motion was a constitutive aspect. Because the motion was uniform, I infer also that a sense of duration of the motion was involved in her concept of length. Finally, because she visually compared the three parts, independently erased the second mark, and then made another estimate of where to place the second mark, I infer that the trace of the motion between its beginning and its end was a third constitutive aspect of length. In fact, she used the records of the motion from one hash-mark to another hash-mark as well as a sense of the duration of the motion in gauging where to put the second hash-mark. I think of Laura's concept of length as abstracted properties of the stick that she introduced in the establishment of the stick. It consisted of moving from one site on the stick to another along with a sense of the duration of movement and its trace. The stick concept included these properties along with the space the trace of the motion occupied. The experience of segmenting the stick in this way was being recorded in her abstract unit items of her number concept, three, which is how I understand a connected number is constructed.

I refer to Laura's scheme for sharing a stick into three equal parts as an *equi-segmenting scheme* in that she used a segment of the stick in sequentially producing the two remaining segments in contrast to Jason's simultaneous partitioning activity. Another distinction between Laura's scheme and Jason's equi-partitioning scheme is that she did not pull the estimate out from the stick and iterate it as did Jason. But in that she seemed to use an image of the first part of the stick in monitoring making the two next parts, she operated *as if* she had pulled the first part of the stick out of the stick and iterated it. This is important to note because, although I can't say that the last two parts were identical to the first part, they were more than simply separated from and unrelated to the first part.

Even though Laura used the arithmetical units of her concept of three to break the stick, her arithmetical units do not account for the material to which they were applied. But she understood before she began her segmenting activity that the whole stick was to consist of three parts and that one of these parts could be used to make the others. I consider the modification in her numerical concept of three that produced a connected number as a functional accommodation of her concept,[3] and it is another initial confirmation of the reorganization hypothesis.

Making a connected number sequence by means of "so many times as long"

Using numerical concepts in constituting unmarked sticks as situations of the concept by means of partitioning or segmenting is crucial in the construction of fractional schemes.

But, this is not the only way we encouraged the children to make connected number sequences. Another way that does not stress partitioning is to use unit segments as discrete unit items to be joined together into connected numbers. Toward this end, we encouraged the children to make connected numbers by constructing meaning for "so many times as long as." The teaching episode from which Protocol IV was abstracted was held on the 14th of October.

Protocol IV: Making sticks two, three, and four times longer than a unit stick.

T: What I want you to start off with today is to make a set of sticks starting with a small unit stick about a centimeter (gauges a centimeter by holding up two fingers about one centimeter apart).
J: (Quickly makes such a stick.)

3 An accommodation is functional if it occurs in the context of the concept or scheme being used.

T: Make a set of sticks starting with one that is twice as long, three times as long, up to … where do you want to go up to? You can do it the quickest way you can figure out using the (computer) buttons you got … can you do it Laura?

L: (Copies two sticks beneath the unit stick, then three sticks beneath those two, then four sticks beneath those three, then four sticks beneath those four. She then more neatly aligns the first four rows of sticks, leaving the last four sticks unaligned.)

T: Now, before you go any further, I want one stick that is twice as long as the unit stick (Laura hadn't joined the sticks together).

J: (Takes the mouse and starts to join the four sticks Laura hadn't aligned.)

T: (To Jason) Maybe you didn't understand what I meant by a set of sticks.

L: (In explanation) You join these together and that would be one, then you put them together and that would be twice as long (the two sticks immediately below the unit stick). And then three times as long … like that.

T: (To Jason) I don't think that was what Laura meant, what you are doing …

J: (Joins all 14 sticks on the screen together.)

T: Ok. Break those apart, Laura, and do what you meant.

L: (Activates break and breaks the 14-stick Jason made into its parts. She then drags the unit stick to the upper left hand corner of the screen and then drags another stick directly beneath it and joins that stick with another stick in the broken row of sticks. She repeats this, making a 3-stick.)

T: Can you explain to Jason what you are doing?

L: (Moves the unit stick with the cursor and then moves the 2-stick with the cursor) that's twice as long (moves the 3-stick with the cursor) that's three times longer than that one, then you can make another line of four, then five, and then have it more and more and more … .

Laura's copying of two sticks beneath the unit stick, then three sticks beneath those two, etc. solidly indicates that "twice as long" and "three times longer" meant to iterate the unit stick twice or three times. That is, Laura considered the stick that Jason made as an arithmetical unit item that she could iterate so many times. Of course, her numerical concepts do not supply all of the necessary meaning for twice or three times longer, so she operated as if she was operating using discrete units to produce specific numerosities (presumably, she anticipated implementing her unit stick an unknown number of times because she said, "and then have it more and more and more").

When Laura joined the sticks together using JOIN, I infer that she used her uniting operation to mentally compound the sticks together into a composite unit. I base the inference on the language she used in explanation to Jason, "You join these together and that would be one, then you put them together and that would be twice as long (the two sticks immediately below the unit stick). And then three times as long … ." What she meant by "that would be one" presumably was that it would be one of several sticks that she made. But she also regarded it as a composite unit item in relation to the unit stick as indicated when she said "that would be twice as long."

The two inferences concerning the operations of iteration and uniting provides the basis for referring to the composite unit sticks Laura made as connected numbers. In making these connected numbers, she did not first engage in partitioning the unit stick and reconstitute it as a connected number. Rather, she produced a connected number much in the same way that she would produce a composite unit of four without engaging in partitioning operations. She treated the unit stick as if it was a discrete unit, and the only difference was in how she used JOIN to implement her uniting operation.

After Laura showed Jason what she meant by making the connected numbers two and three (hereafter, rather than say "connected number three," I use "3-stick" for simplicity), and then saying that she could continue on in the same way for four and five, Jason finally understood what the teacher intended as indicated by his subsequent activity. He joined four of the remaining copied unit sticks together and placed the 4-stick he made under the 3-stick Laura had made. There were only four copied unit sticks left, so he made another copy and joined the five copied unit sticks together and placed them under the 4-stick. To make the next stick, Jason copied the 5-stick and the unit stick and joined them together. To make the 7-stick, Laura then copied the 6-stick and the unit stick and joined them together. After Jason made the 6-stick, the teacher asked Laura if she wanted to carry on and what the next stick would be. Laura said, "seven," so the teacher asked her to find a quick way to make the seven. Laura asked, "Can it be the same one he did?" meaning if she could make it the same way. From her comments and action of making the 7-stick by joining a copy of the unit stick to the 6-stick, we can infer that she re-presented Jason's actions and abstracted the process of producing a number by taking the previous number and adding one. She saw in Jason's actions a general way of operating because she operated analogously for the 8-stick and commented to the teacher that to make the next stick, she would add one more. She was obviously aware of what she was doing and we can say that for both Laura and Jason, a connected number was related to its successor by the relation of "one more" in a way quite analogous to how a whole number was related to its successor. For this reason, I infer that both children had constructed the possibility of producing the next connected number in a sequence of connected numbers, and hence, had constructed a *connected number sequence*.

In the case of the 9-stick, Jason changed from simply adding one more stick to the 8-stick and made copies of the 6-stick and the 3-stick and joined them together, confirming that the sticks he made were indeed connected numbers. To make the 10-stick, he used the 7-stick and the 3-stick, which is further confirmation. Laura then used the 5-stick and the 6-stick to make the 11-stick. In this way, the children used their adding schemes to make successive connected numbers. After making the 14-stick, the teacher asked the children to make what was to be the last and final stick of the series using only one kind of stick to make the 15-stick. Jason copied the 3-stick five times and joined them together and Laura said she was going to use the 5-stick. That is, they knew that five iterated three times or three iterated five times would produce 15.

In that Jason did not initially understand the intention of the teacher, but operated very powerfully upon recognizing Laura's language and actions involved in making sticks twice as long and three times longer than the unit stick is solid indication of generalizing assimilation.[4] His use of his numerical schemes were indeed blocked by the language "twice as long" and "three times as long" and by how to make sticks that were of those sizes. He did know how to make a stick by drawing and by copying a stick, and he knew how to join sticks together, but he didn't know how to use these operations to make the sticks requested by the teacher. Nevertheless, once he recognized what Laura was doing and why, there were no major modifications necessary in his numerical schemes for him to operate as he did. He operated powerfully and smoothly as if his operating referred to discrete items that couldn't be joined together physically. Laura, of course, experienced no blockage of the nature experienced by Jason and also operated powerfully and as if she was operating with discrete items. So, if a generalizing assimilation was involved in her case, it was immediate.

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4 An assimilation is generalizing if, first, the concept is used in situations that contain sensory material that is novel for the concept, and if there is an adjustment in the use of the concept (cf. Steffe & Wiegel, 1994; Steffe & Thompson, 2000b).

An attempt to use the children's multiplying schemes in the construction of composite unit fractions

Based on Jason's and Laura's use of their multiplying schemes to make a 15-stick, we conjectured that the children could use this and a related scheme in the construction of composite unit fractions. For example, if Laura knew that she was to copy the 5-stick three times to make a 15-stick, then we conjectured that she could establish the 5-stick as one-third of the 15-stick. If the conjecture proved to be viable, then it would be possible that the children could establish that one-third and five-fifteenths, say, were commensurate fractions. In establishing the two fractions as commensurate, we surmised that the children would need to maintain an awareness that the five-to-fifteen comparison produced a segment the same length as the one-to-three comparison. This would mean that the children would need to maintain an awareness of the structure of the unit containing three units of five and operate on the units of this given structure as well as on the units of one that the three units of five contained. If the children could engage in such operating, then a path to commensurate fractions would be established that would be quite powerful.

From our work with the two children in the first year of the teaching experiment, we knew that they had constructed what I call a *units-coordinating scheme* (Steffe, 1992). In general, I think of units-coordinating as the mental operation of distributing a composite unit across the elements of another composite unit. For example, if an occasion should arise where a child forms a goal of finding how many pennies in all could be placed into five bags if a pair was put into each bag, the operation of units-coordinating could be used to solve the problem. The child posits a composite unit of five and a composite unit of two and mentally inserts the composite unit of two into each of the individual units of the composite unit of five prior to any other acting or operating. The result of the units-coordination permits the child to experience a unit containing five units of two that is not simply a sensory-motor experience. This organized experience is what I regard as a *situation* of the scheme. The *activity* of the scheme in the case of the example is to count by two five times to specify the numerosity of the individual units contained in five units of two, and the *results* of the scheme is the experience of the immediate past counting activity along with its result.

Other than a units-coordinating scheme, I constructed a dividing scheme that I called an *equi-portioning scheme* based on the dividing activity of Jason and Patricia during their fourth grade in school. The child again posits two composite units, say a unit of 21 and a unit of seven. The child's goal is to find how many of the 21 units would fit into each of the seven units if an equal number were placed into each of the seven. The activity of the scheme is to posit a portion of the 21 units and iterate that portion seven times in a test to find if it works. The activity of the scheme is quite similar to that of the units-coordinating scheme, and it could be said that the equi-portioning scheme contains the units-coordinating scheme as a subscheme in that the child distributes the posited portion across the seven units prior to initiating the activity.

Conflating units when finding fractional parts of a 24-stick

Four teaching episodes were held between those held on the 14th of October and the 2nd of December that were devoted to the children using their equi-portioning scheme in the context of connected numbers. On the 2nd of December, the teacher asked a fractional question in the context of the children using their equi-portioning scheme in a test of our conjecture that children could use their multiplying and dividing schemes in the construction of composite unit fractions. The children started the teaching episode by making a unit stick (1-stick), and then making all the sticks from the 1-stick through the 10-stick and erasing all of the marks on the sticks. There was now a graduated collection of ten unmarked sticks on the screen. The teacher then asked the children to make a 24-stick using the sticks of the graduated collection as a preliminary to a fraction task he had planned. Jason selected the unmarked 3-stick and iterated it eight times to make a 24-stick marked into eight parts. Laura then selected the unmarked 6-stick and iterated it four times to make a 24-stick marked into four parts.

The teacher then posed his fraction task. After asking the children to hide their eyes, he made a 24-stick using the unmarked 3-stick. The 24-stick was marked into eight parts each of which was unmarked.

Protocol V: Finding the fractional part a 3-stick is of a 24-stick.

T: I used one of the sticks. Which one did I use and what fraction is it of the 24-stick?
L: It is either the two or the three.
J: Three. It's the three (Laura agrees).
L: And the fraction is three-eighths!
J: Three-eighths (in agreement).
T: (To Laura) you said two or three and (to Jason) you said three. How did you find out?
J: Same.
L: They look the same (referring to the unmarked 3-stick and to the eight parts of the 24-stick).
T: And how did you find out (to Jason)?
J: I went 3, 6, 9, 12 and ended up to 24. And I know it would be a three because I used a three.

Laura said "three-eighths" to indicate the fractional part the 3-stick was of the 24-stick and Jason agreed with her. The children focused on the numerosity of the 3-stick when saying "three-eighths" rather than on the stick as a composite unit. The 24-stick was marked into eight equal parts and it was in full view of the children. Still, they said "three-eighths." So, the teacher continued on, investigating whether this would recur in other cases.

Protocol V: (Cont.)

T: Jason, close your eyes. Laura, pick any other way to make the 24-stick. Any way you like.
L: (After approximately 14 seconds during which she subvocally uttered number words while looking at her hands resting underneath the table, she makes a copy of the 2-stick and repeats it 11 times to make a 22-stick. She then makes a correction by copying the unmarked 2-stick and joining the copy to the 22-stick.)
T: Ok, Jason! That's the problem. You need to say which stick she used and what fractional part it is of the 24-stick. And you need to verify it.
J: (Uncovers his eyes. Copies the 2-stick into the ruler and measures Laura's stick, "12" appears in the number box.) She used the two.
T: And? What fractional part is it?
J: (Sits quietly without answering.)
T: Remember what we call it? What does this number "12" tell you about the name of the fraction?
J: How many times you click!
T: So, what would you call the fraction?
J: Two-twelfth!
T: One-twelfth.
J: One-twelfth (almost simultaneously with the teacher).

T: Laura, you close your eyes. (To Jason) you are going to build a 24-stick but do not use the two or the three.
J: (After a short pause, makes a copy of the 6-stick and repeats it four times.)
T: (After asking Jason to think of what part the 6-stick is of the 24-stick) alright Laura, here is the problem (points to Jason's 24-stick.) Find which stick it is and what fractional part it is.
L: (After visual inspection) he used the 6-stick four times.
T: How did you know it?
L: I looked at this stick (the 6-stick), and then I looked at it (Jason's 24-stick).
T: Oh! Like before with the two and three and you tried to see which one?
L: (Nods her head "yes").
T: (Points to the first part of Jason's 24-stick) And what do you call the 6-stick here?
What part of the 24-stick is it?
L: Six-fourths.
T: Six-fourths. How do you verify? (To Jason) is this the name of the part she used?
J: The name of the part?
T: (Asks Jason what part the 6-stick is of the 24-stick) what fractional name do we give it? How do you verify it?
L: Six-fourths.

Laura knew that Jason repeated the 6-stick four times and that it produced a 24-stick. However, whether she was aware that iterating the 6-stick four times produced a 24-stick partitioned into four equal parts, one of which was one out of four equal parts of the 24-stick, is problematic. There seemed to be a lacuna in her reasoning. Although she could make composite units, she didn't seem aware of the structural relations among the three levels of units she produced. The same lacuna seemed present in Jason's thinking as well, because he said "Two-twelfth!" in the immediately preceding task.

There is solid indication in the last of the next two tasks of the teaching episode that Laura was aware that the number of iterations of a stick was necessarily equal to the number of parts produced. Jason repeated the 1-stick 24 times to make a 24-stick. He did not erase the marks, and Laura almost immediately said that he used the 1-stick 24 times and that it is one-twenty fourth. So, when the stick iterated was a unit stick, she operated as we hoped she would operate when the stick iterated was a composite unit stick.

Laura saying "one-twenty fourth" does indicate that she was aware that the number of times that Jason used the 1-stick was necessarily equal to the number of parts of the stick produced. Moreover, she seemed to be aware of a comparison between one part and the twenty-four parts. So, when she said "six-fourths," I infer that she was indeed aware of making a part-to-whole comparison. But saying "six" rather than "one" constitutes a conflation of units that I believe was a necessary consequence of her current operations.

The case of a 12-stick as one-half of a 24-stick

Indication of the status of the children's part-to-whole operations was provided in the task immediately following the task of Protocol V. Laura copied the 12-stick and repeated it to make the 24-stick and the teacher asked her to prove to Jason that she used the 12-stick.

Protocol V: (Second cont.)

T: How would you prove to him that you used the 12? (speaking to Laura).
L: That I put the 12 into the measure and I measure it and it came out to be two times.
T: And it is the 24-stick, so?

L: Measure it two times. *T:* It needs to be the 12? *J:* It has to be.
T: It has to be? If you use the 12 twice you get 24. Why does it have to be?
J: Because there is no other way, only 12 plus 12 is 24.

Jason then used TIMA (sticks), to check what was already certain to him: there is only one way to express the relation between the 12 and the 24 because, for him, "only 12 plus 12 is 24." When asked what fraction the 12-stick was of the 24-stick, Jason said "one-twoth!" In that he knew that Laura had used the 12-stick, he obviously focused on the 12-stick as one stick and made a one-to-two comparison. But this was because of the special nature of one-half.

Laura's "proof" that she used the 12-stick does indicate that she was aware of the two 12-sticks as units belonging to the 24-stick, but yet as units apart from the 24-stick. But saying "I measure it and it came out to be two times" by itself wouldn't be very convincing. However, in an earlier teaching episode, she said that measuring an unmarked 10-stick using an unmarked 5-stick would be two and explained as follows: "Because the five goes in one time and then another time (touches the tips of her two forefingers together simulating the placement of the 5s in the 10-stick) and then five plus five is ten." This explanation convinces me that she regarded the 5-stick as a unit apart from the 10-stick and that she regarded the 10-stick as being comprised of two 5-sticks.

Necessary errors

I interpret the children's answers of "three-eighths" rather than "one-eighth," "two-twelfth" rather than "one-twelfth," and "six-fourths" rather than "one-fourth" as *necessary errors* rather than as errors due to a simple misinterpretation of the situation. An error (from the observer's perspective) is necessary if it occurs as the result of the functioning of a child's current schemes. I have argued that both Jason and Laura, 12 and 24 for Jason and 5 and 10 for Laura, did disembed a numerical part from a numerical whole and conceive of the whole as consisting of its two parts. This was, of course, in the case of two parts of a whole, and in this case Jason said that 12 is "one-twoth" of twenty-four. So, in that case where Jason conceived of 12 as a composite unit, and of 24 as consisting of two such units, he made an appropriate one-to-two comparison. In fact, both children made such numerical comparisons in the teaching episode held on the 11th of November. This is certainly an indication of the kind of reasoning which I have in mind that would occur upon a reorganization of the children's equi-portioning scheme for composite units.

As I have already commented, a situation of the equi-portioning scheme involves two composite units one of which serves as a container containing several other unit containers, and the other of which serves as a source of units to be inserted into the unit containers. In the equi-*partitioning* scheme for a unitary item that I explained in the context of Protocol II, the situation involved only one composite unit that was used in partitioning the unitary item into equal parts. Reasoning analogously, if we posit an equi-*partitioning* scheme for a unitary item that is a connected number, then the situation would still involve a composite unit that would be used as a template for partitioning the connected number. In this case, rather than serve as a container containing several other unit containers, the second composite unit would be used as *an operation of partitioning* the connected number.

That the children said "three-eighths" rather than "one-eighth," "two-twelfth" rather than "one-twelfth," and "six-fourths" rather than "one-fourth" could be interpreted simply as their whole number knowledge temporarily interfering with their making the appropriate unit comparisons. In fact, in contrast to the reorganization hypothesis, there is a widespread and accepted belief that whole number knowledge interferes with the learning of fractions (Post, Cramer, Behr, Lesh, & Harel 1993; Lamon, 1999). The belief portrayed by these researchers is similar to a comment made by Streefland (1991) in a detailed report of children's fractional knowledge: "But the only alarming ailment is the following one, namely, the temptation to deal with fractions in the same manner as with natural numbers" (p. 70). Streefland believed that we must focus on forming a powerful concept of fractions which is resistant to whole number distractions. According to Streefland (1991), "we must not only focus on producing fractions, but also on grounded refutations of such misconceptions, or simply, on overcoming these misconceptions" (p. 70). However, to say that the numerosity of the 3-stick interfered with the children establishing the 3-stick as one-eighth of the 24-stick does not take assimilation into account. The children were using their equi-portioning schemes to establish what stick the other child used to make the 24-stick, so the part-to-whole comparisons that they made were constrained to the nature of the structure of the results produced by that scheme. My argument is based on the consistency of their "errors" and by Jason's appropriate one-to-two comparison in the case of using a 12-stick to make a 24-stick. He treated the two 12-sticks as two unit sticks apart from the 24-stick as well as comprising the 24-stick, and this permitted him to compare a 12-stick to a 24-stick in a one-to-two comparison.

A simultaneous partitioning scheme

The experiment to explore whether the children could use their equi-portioning scheme in the establishment of composite unit fractions essentially failed in that the children, although they could find what stick repeated, say, six times makes the 24-stick, they conceived of it as four-sixths of the 24-stick rather than one-sixth of the 24-stick. Because I construed this as a necessary error, we shifted from our attempts from bringing forth the equi-portioning scheme as a cognitive mechanism for the construction of composite unit fractional schemes, to the children's partitioning and segmenting schemes involving only continuous unitary items as elements of the partition or segmentation. To illustrate how the children could use their numerical concepts through ten in partitioning and segmenting, I select a protocol from the teaching episode held on the 7th of December.

Protocol VI: Drawing a stick that is one-tenth of another stick.

T: Can each one of you draw one-tenth of that stick? The one who wins will be the one that will be closer.
L: (Draws her estimate) Right there!
J: (Looks at the screen for some time and draws his estimate.)
L: That's the same!
J: No it isn't, no it isn't!
L: Ok! I will go first here! (She repeats the stick 10 times and it is too long.)
J: (Even though his estimate is longer that Laura's, he still repeats it 10 times to check.) Oh gosh! (Both children giggle.)
T: You want to try one more?
L: I want to try it one more time!
T: One-tenth, all right!
L: One-tenth, one-tenth! Ok! This is my color! Ok that was too long … ok! That long! (draws her estimate).
J: (Draws his estimate, both children laugh.)
L: (Repeats her estimate reciting the times out loud, and the estimate is very accurate) Just about!

T: Very close!! Let's see Jason. That's very nice!
T: (Speaking to Jason) What do you think yours is? Too short or too long?
L: Too short!
J: (Repeats his estimate which is shorter than Laura's and so produces a shorter stick than the unit stick. The children giggle.)
T: All right a little bit too short!

The children became deeply engaged in the task and expressed pleasure at making an estimate by drawing a stick and then testing their estimates by using REPEAT. The initial estimates of both children were closer to one-eighth and one-seventh of the unit stick (Laura and Jason, respectively) and Laura's second estimate was uncannily accurate. Thus, the children used the iterative aspect of the units of their connected numbers to test their estimates by iterating them ten times and comparing the results against the original stick. In this way, the children segmented the original stick using their estimate.

That Laura made such an uncannily accurate estimate on her second trial may have been fortuitous. So, in the next teaching episode, held on the 8th of February of the next calendar year, the teacher posed a task involving sharing a stick into eight equal parts. Other than serving as a check of Laura's as well as Jason's estimates, the teacher wanted to explore whether Laura's use of iteration in Protocol VI was specific to the estimation task. Sharing tasks emphasize fragmenting rather than segmenting, so Laura may have no reason to iterate in order to verify her estimates of an equal sized part. The initial task was, after Jason drew a segment the same length as a Snickers Candy bar, to share it equally among eight people. The teacher imposed the constraint that they could make only one mark.

Protocol VII: Sharing a candy bar among eight people by making only one mark.

T: (Counts all the persons in the room aloud: 1, 2, 3, 4, 5, 6, 7, 8.) Your first task is to share this candy bar among these people. But use only marks. Remember you can move marks. But mark only the share of one person and use that to create all the shares. Just one mark for eight people. Go ahead.
J: (To Laura) go ahead.
L: (Takes the mouse and activates Marks) But we can use a lot of marks to … .
T: Use one mark, if it will not come out as a fair share then you can use another, but try to make it as close as you can in the beginning.
L: (Activates Marks again and tries to estimate where to put the first mark. She makes an uncannily accurate estimate. The mark she makes on the stick is apparently one-eighth of the unit stick, but she is yet to verify her estimate.)
T: You know, we can still play with the screen. Remember PULL PARTS and REPEAT? (indicating to Laura that she isn't done).
L: Ok, there's … .
T: You remember, PULL PARTS and REPEAT.
L: So, can I make another mark?
T: No, no, just one mark. Now see if its a fair share.
L: (Seems confused and looks for a button to use in her microworld.)
T: Do you want to pull the part first?
L: Ok. (Activates PULL PARTS and pulls the greater of the two parts from the marked stick. She sets the cursor over the smaller piece) Do I do this piece too (the marked piece that she estimated as the share of one person)?
T: Which one do you want to use to check to see if its one-eighth?
L: Umm, this one (points to the 7/8-stick she pulled out from the marked unit stick)?

T: … Now how can you tell that this (pointing to the 7/8-stick) is exactly one-eighth?
One-eighth of the candy bar. This is the candy bar (opening his hand over the length of the original stick).
L: I don't know.
T: (To Jason) Jason, do you have an idea?
J: (Nods "yes" and takes the mouse, drags the 7/8-stick to the top of the screen, then pulls Laura's estimate from the marked stick.)
T: Can you tell Laura what you are going to do?
J: I'm gonna' … pull one of these (points to the 1/8 part of the marked unit stick) and put it under there and see if … .
L: (Enters Jason's talk, nodding "yes") and repeat.
T: Ok.
J: (Repeats the 1/8-stick eight times until it reaches the length of the original stick. The resulting 8/8-stick seems to be exactly the same length as the unit stick.)
T: Wow, Wow, Wow! Laura you made it so quickly!! One, two, three, four, five, six, seven and eight. I don't believe it! Isn't that great! It's really good! (The children then marked the unmarked original stick using the 8/8-stick as a template.)

The accuracy of Laura's estimate should not be regarded as fortuitous. Her estimate of one-tenth in the 7th of December was also uncannily accurate on her second trial and there were other occasions where she made similar accurate estimates. Her comment "But we can use a lot of marks to ... " should be considered as indicating that she visualized marks on the stick so that eight parts would be formed. She could then accurately gauge the length of one of the parts.

Although Laura made an uncannily accurate estimate of one-eighth of the stick, she did not independently use PULL PARTS and REPEAT to verify the estimate. Pulling the 7/8-stick from the marked unit stick, when coupled with her choice of the 7/8-stick after the teacher asked her which of the two parts she wanted to use to check to see if its one-eighth, indicates that she intended to continue on, marking the 7/8-stick. Unfortunately, the teacher asked her how she could tell if the 7/8-stick is exactly one-eighth, and his question closed off any further actions she may have taken with the 7/8-stick.

Laura definitely could use her number concepts up to ten as templates for partitioning blank sticks in the true sense of a partitioning. In fact, in Protocol VI, it is plausible that she used the composite unit, ten, as a partitioning template in making her estimate. In that case, she would *simultaneously* project the units of her composite unit into the blank stick and experience the parts as co-occurring. In Protocol VII, this partitioning activity seemed to close off her need to verify the part she marked off by pulling the part from the original stick and iterating it eight times to make a test stick. The operation of iteration unquestionably was available to her as indicated by Protocol VI. But, in that case her estimate was not a part of the stick of which she was estimating a part, whereas is Protocol VII, her estimate was a part of the original stick. Although iterating, partitioning, and disembedding were operations of her number sequence, she seemed to only use partitioning in Protocol VII. That is, she used her connected number concept, eight, to project units into the stick. Jason, on the other hand, disembedded the part of the stick Laura had made and iterated that part eight times in an attempt to find if it was indeed 1/8th of the unit stick. In that Laura made such uncannily accurate estimates when partitioning a blank stick into up to ten parts, I refer to the scheme she used as a *simultaneous partitioning scheme* to distinguish it from Jason's equi-partitioning scheme. Her uncanny ability to make accurate estimates seemed to suppress the use of the disembedding and iterating operations because she felt no need to verify her estimates.

The lack of a unit fractional scheme

Partitioning schemes are schemes for making fair shares rather than fractional parts of wholes. So, we focused intensively on bringing forth Jason's and Laura's use of their partitioning schemes in the construction of fractional schemes while they were in their fourth grade. We made extensive use of tasks involving sharing and the estimating of shares, and integrated fractional language into our interactions with the children. I now discuss the modifications that were involved in the children constituting their partitioning schemes as unit fractional schemes and the constraints we experienced. In a test of the limits of the children's partitioning schemes in the teaching episode held on the 17th of February, the teacher posed the tasks of Protocol VIII.

Protocol VIII: Making a stick so that a given stick is five times longer than the stick to be made.

T: Ok! Let's draw a stick … [he draws a two inch stick]. Can you show me a stick that is five times longer that the stick?

L: (Activates COPY and makes a copy of the teacher's stick. She then activates REPEAT and makes a 6-stick instead of a 5-stick because she interprets "five times longer" as "five more".)

T: (To Jason) can you show me a stick so this one (points to an existing stick in the screen that Laura had drawn just before the teacher drew the two inch stick) is five times longer than the stick you show me?

J: WHAT?

T: Ok! This is my stick (pointing to the existing stick). I want you to make a stick such that mine is five times longer than yours.

J: Five times longer?

T: Yes. Mine will be five times longer than yours.

J: (Makes a copy of the stick, activates PARTS, dials to "10," and clicks on the copy, marking it into ten equal pieces. He then breaks the stick using BREAK and joins the first five pieces back together. He then drags the five extra pieces into the TRASH.)

T: Ok. Mine is five times longer than yours? Can you show me that?

J: Mmm Mmm (yes). (Repeats the 5/10-stick he made and places the resulting 10/10-stick in the middle of the screen. He then places a copy of the unmarked original stick directly above it and then places the 5/10-stick above the unmarked original stick aligned at the left endpoints as shown in Fig. 4.)

T: How many times did you repeat when you did it?

J: Two, one time.

T: So when you repeat it … is this five times longer?

J: This one is (pointing to the top 5/10-stick) but that one no, this is ten (pointing to the bottom 10-stick).

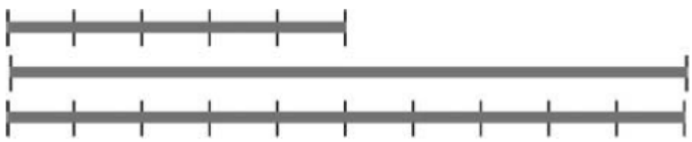

Fig. 4. Jason's attempt to make a stick so that a given stick is five times longer than the stick.

It is quite significant that Jason did not confuse the question the teacher asked him; "Can you show me a stick so this one is five times longer than the stick you show me?" with the question the teacher asked Laura: "Can you show me a stick that is five times longer than this one?" To solve the problem as the teacher intended, it would have been necessary for Jason to posit a *hypothetical stick* such that repeating that stick five times would be the same length as the teacher's stick. This would require the operations of partitioning and iterating be implemented simultaneously rather than sequentially. That is, he would need to not only posit a hypothetical stick, but also to posit the hypothetical stick as one of five equal parts of the teacher's stick that has been already iterated five times and see the results of iterating as constituting the teacher's stick. This is a *composition* of partitioning and iterating and I refer to it as a *splitting operation*.[5]

A comparison of two concepts of splitting

I now turn to Confrey's (1994) idea of splitting and show how my analysis of splitting and of the equi-partitioning scheme for connected numbers are involved in how she defines splitting. According to Confrey (1994):

> In its most primitive form, *splitting* can be defined as an action of creating simultaneously multiple versions of an original, an action often represented by a tree diagram. As opposed to additive situations, where the change is determined through identifying a unit and then counting consecutively instances of that unit, the focus in splitting is on the one-to-many action. Closely related to this primitive concept are actions of sharing and dividing in half, both of which surface early in children's activity. Counting need not be relied on to verify the correct outcome. Equal shares of a discrete set can be justified by appealing to the use of a one-to-one correspondence and in the continuous case, appeals to congruence of parts or symmetries can be made. (p. 292)

In further elaboration of a split, Confrey (1994) commented that: "A split is an action of creating equal parts or copies of an original" (p. 300). She clearly considered splitting and sequencing to be independent in their origins and this would seem at the outset to countermand the reorganization hypothesis. Kieren (1994) seemed to agree with Confrey that the separation between splitting and number sequences is fundamental even though he identified a different basis for the separation than did Confrey. "In fact, Confrey's splitting analysis seems to point to actions as opposed to number scheme research ... , which points to units of units" (pp. 391–392). My claim is that the splitting operation, as I defined it, can be interpreted as the *initial split* in Confrey's definition of a split, and that the equi-partitioning scheme for connected numbers can be interpreted as a split of a split in her definition.[6]

When a child has constructed the splitting operation as I have defined it, the child can focus on the continuous unitary item that is to be partitioned, which is the unit item implied by "one" in "one-to-many," as well as on the number of parts into which the child intends to partition the continuous unit, which is the "many" in "one-to-many." This is made possible by the child being aware that the "one" constitutes the "many," and vice versa, without further action or operation. That is, the child is aware of the unit structure of the composite unit that is projected into the continuous unit in such a way that it both contains and partitions the continuous unit. This produces an initial experience of one-to-many.

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5 See Sáenz-Ludlow (1994), for an analysis of the constructive power of a child, Michael, whom I infer had constructed this operation.

6 I recognize other one-to-many mappings that do not involve partitioning.

But the child can do more in splitting. If the child focuses on the elements of the composite unit into which the continuous unit is to be partitioned, the child can move back again to the unit structure. Moreover, the child can disembed any subcollection of elements from the composite unit and regard it as a composite unit in its own right. This establishes the classical numerical part-to-whole operation that serves as a fundamental operation in the construction of fractional schemes.

Finally, the child can use any singular part of the partitioning in iteration to establish a connected but segmented unit equivalent to the original continuous unit, which is a realization of the one-to-many relation by starting with the part. The child can also use any singular part of the partitioning in iteration to produce a composite unit of elements of numerosity less than the numerosity of the composite unit used in partitioning and then compare that composite unit with the original composite unit. In this way, a child can establish meaning for, say, 4/7 as 1/7 four times or 4 × 1/7. The child can also compare 4/7 with 7/7 and understand 4/7 as four units out of seven units. These are all crucial operations in the construction of fractional schemes that one-to-many mappings alone do not account for.

Rather than focus only on units of units as an end in themselves, as Kieren (1994) suggests, I also focus on the operations that produce them. This permits the use of units of units in my analysis as templates for partitioning actions. In the case where the unit being partitioned is a connected number of numerosity greater than 1, rather than split the whole of a connected number, say, three, into, say, eight parts, the child might distribute the partitioning operation across each of the three parts and establish that each of these parts produced is one-twenty fourth of the original connected number. The child might then select one-twenty fourth from each of the three partitioned parts eight times and establish that each selection produces three-twenty fourths of the original connected number. These operations were far beyond Jason and Laura while they were in their fourth grade. There was no indication that they had constructed even the basic operation of partitioning the whole of a connected number, such as three into eight parts. Nor was there any indication that they had constructed the splitting operation (they couldn't even make a stick such that a given stick was twice as long as the one they were to make). The lack of these operations is the main reason why I consider their answers of "three-eighths" rather than "one-eighth," "two-twelfth" rather than "one-twelfth," and "six-fourths" rather than "one-fourth" as *necessary errors*. As I see it, partitioning the elements of a connected number into composite units of equal numerosity is necessary in splitting a split, so I would surmise that Confrey's notion of splitting would run into the same kind of constraints that I experienced in the case of Jason and Laura in our attempts to use their multiplying and dividing schemes in construction of composite fractional units, such as a 3-stick that was one-eighth of a 24-stick.

The partitive unit fractional scheme[7]

Jason's choice of ten parts in Protocol VIII does indicate that he was aware that the stick he was to make had to be shorter than the teacher's stick. It also indicates that he was aware that the stick he was to make was a part of the teacher's stick. Choosing ten as a partition of the original stick and then using this partition to make a 5-stick also indicates an intuitive awareness that the stick he was to make should be embedded in the unmarked stick. Although a sense of embeddedness is essential in the construction of the composition of partitioning and iterating, Jason partitioned the original stick into ten parts because he

7 See also Tzur (1999) for a discussion of this scheme.

needed a target number to work with. He could then give meaning to "five times longer" by taking one-half of the 10-stick he made and this satisfied his goal of making a stick that could be iterated five times. I regard Jason's attempted solution as creative mathematical activity as well as indicating that the splitting operation was within possibility for him. But he was yet to construct the composition of partitioning and iterating, which is crucial in establishing a unit fraction. Nevertheless, unit fractional language had meaning for both children.

In the following continuation of Protocol VIII, it is possible to understand what "one-tenth" meant for both Jason and Laura.

T: What do you mean when you say 10?
J: Well … It's the same size as … yours (pointing to the copy of the original stick).
T: So, what is this (pointing for one of the pieces of the 10-stick produced by Jason), what would you call it? How much is it … pull that part … .
J: One-half!
T: Pull that part. Pull one small part of your parts.
J: (Jason pulls one of the parts of his 10-stick.)
T: Ok! This is your part for the time being.
J: One-tenth.
T: So this part is one-tenth of mine?
J: Uh-huh (yes).
T: What do you say? He says that this piece is one-tenth of mine. Is that ok? (speaking to Laura).
L: Yeah.
T: Why do you think is one-tenth? [speaking to Jason].
J: Because this is one out of 10 little pieces (holds his left thumb and forefinger about one centimeter apart indicating a little piece).
T: Ah! I see. Now if you repeat this one 10 times … .
J: Which one?
L: This one (Laura is following the dialog and knows what the teacher wants Jason to do).
T: What would you get? *J:* Which one do I repeat? *T:* Yours.
J: Mine?
J & *L:* 10 times?
L: Ten-tenths!
T: Ten-tenths! What do you say? (speaking to Jason).
J: Ten-tenths?
T: Do you want to do it?
J: Repeat that one?
T: Your piece, yes.
J: (Jason repeats the 1/10-stick 10 times.)
T: So how much is ten-tenths, Laura?
J: The whole stick (note that Jason is continually aware of the whole stick).
L: Ten of those little sticks.
T: Did you hear what Jason said? That ten-tenths is the whole stick? Do you agree with it?
L: I guess so.
T: Why?
L: I don't know.
T: Why do you say that ten-tenths is the whole stick?
J: Because it is 10 little pieces and it is a how long the whole stick is. So one whole stick is 10 pieces of those little ones.
T: Did you get it? (speaking to Laura).
L: Yes.

It is now possible to understand what "one-tenth" meant for Jason as well as for Laura. In his explanation of why the teacher said "one-tenth" — "Because its one out of ten little pieces (holds his left thumb and forefinger about 1 cm apart indicating a little piece)," the language "out of" is a key. It indicates that he regarded one little piece as one unit part out of the ten unit parts from which it originated. So, I attribute a part-to-whole relation to Jason as constituting his meaning of "one-tenth." He also regarded the ten unit items as constituting the original stick, and he knew that iterating the one-tenth part would produce the ten unit items, and thus, the whole stick. So, one-tenth was an *iterable fractional unit*. I call the scheme he used to establish one-tenth as a *partitive unit fractional scheme* to emphasize that the dominant purpose of the scheme was to divide the connected number, one, into so many equal parts, take one out of those parts, and establish a one-to-many relation between the part and the partitioned whole. The iterative aspect of the scheme served in justifying or verifying that a unit part of the connected number, one, was one of so many equal parts.

The partitive unit fractional scheme differs from the equi-partitioning scheme in the explicit numerical one-to-many comparison (one-to-ten) and in the explicit use of fractional language "one-tenth" to refer to that relation. It may seem as if Jason had constructed the splitting operation when he said, "Because it is ten little pieces and it is how long the whole stick is. So one whole stick is ten pieces of those little ones," in explaining why ten-tenths was the whole stick. Based on this comment, there is no doubt that he understood that the ten pieces of "those little ones" comprised the whole stick. He also regarded the length of the stick as ten little pieces. So it would seem indeed that he had constructed the splitting operation. However, there is no indication that he had constructed a multiplicative relation between the whole unpartitioned stick and one of its hypothetical parts. In this relation, he would need to be simultaneously aware of the whole stick as a unit stick and of a hypothetical part of the unit stick such that the unit stick consists of, say, ten iterations of the hypothetical part. I emphasize "hypothetical" because the child must produce an image of *some* stick that is considered to be a part of the unit stick, and mentally set it in relation to the unit stick such that a multiple of the hypothetical part constitutes the unit stick, prior to any observable action. Instead of producing such a stick and consider it as defining a partition of the unit stick in the beginning of Protocol VIII, he instead chose to partition the unit stick into ten parts to establish a target number of parts. It was as if he needed two specific numbers so he could establish one of them as five times the other. Nevertheless, it is consequential that both he and Laura knew that if one of ten little pieces was iterated ten times, the result would be ten-tenths of the stick. This is basic, and I regard it as essential for a scheme to be called a unit fractional scheme.[8]

Establishing fractional language for multiple parts of a stick

Based on the way both Laura and Jason produced "ten-tenths" in the continuation of Protocol VIII, we proceeded to explore their production of fractional language as a consequence of the functioning of their schemes. Our hypothesis was that the children's

[8] There is some doubt whether Laura knew that ten-tenths constituted the whole stick in the continuation of Protocol VIII. But, she was the first to say "ten-tenths" for the result of repeating one-tenth ten times. Still, she didn't seem to explicitly realize that a 10/10-stick constituted the whole stick.

partitive unit fractional schemes would both enable and constrain the language they produced.

Operating iteratively in the production of fractional language

In exploration of how the children's partitive unit fractional scheme would enable the establishment of fractional language, he teacher posed the task of Protocol IX. This protocol was extracted from the teaching episode held on the 7th of March.

Protocol IX: The production of fractional language.

T: The birthday party was going on when I came into the class. What will be the share that three people will get (of the birthday cake)? … On the cake, show me the piece that, let's say, that three of us will get.
J: (Pulls the first part from the stick and repeats it to make a 3/24-stick.)
L: That's what I was going to do.
T: Alright, Laura, if that was what you were going to do, tell me how much the share of three of us together will be out of the whole cake.
L: Three-twenty fourths!
J: (Nods "yes.")
T: So, now, let's say another three people came in. Can you show me the part that will be?
J: Three more?
T: Three more. So we have altogether six. We still have 24 but we want to see the share of six of us.
L: Six of us. (Activates REPEAT and clicks on the 3/24-stick) Oh! There.
T: That's it? *L:* That's it. *T:* Explain.
L: Ok. I repeated it, and now there are six people that could get their … , Ok (points to each piece with the cursor) 1, 2, 3, 4, 5, 6.
J: (With Laura) their share.
T: All right. So how much is this out of the whole cake?
J & L: Six-twenty fourths.

It is quite important that Jason did not simply partition the stick into 24 parts and then fill three of them with a color. Rather, he used the operations of his partitive unit fractional scheme when he partitioned the stick into 24 parts, pulled one part out and then iterated it three times to produce the share of three people. Laura saying "That's what I was going to do." and then saying that the stick Jason made for the share of three of the twenty-four people was three-twenty fourths of the stick, when coupled with her actions of repeating the 3/24-stick to generate a share for three more people, does indicate that her actions with the sticks were representative of her reasoning in the sharing of a birthday cake among twenty-four people.

It is also important to note that the teacher asked, "All right. So how much is this out of the whole cake?" before either of Jason or Laura said, "Six-twenty fourths." That is, although Laura replicated the 3/24-stick to make the 6/24-stick, the children didn't seem explicitly aware that six-twenty fourths was twice three-twenty fourths. They knew that six was twice three, but the 6/24-stick was named "six-twenty fourths" because it was six out of twenty-four pieces. This seemed to be the case for the 3/24-stick as well. The 6/24-stick and the 3/24-stick were *results of operating*, and the children constituted these sticks as fractional parts of the original stick by comparing them to the original stick. Given the iterative property of their partitive unit fractional schemes, it may seem surprising that the children compared the two sticks with the original stick in order to give them a fractional meaning.

They did realize that "ten-tenths" referred to ten of the one-tenth parts, and likewise they realized that "three-twenty fourths" referred to three of the one-twenty fourth parts. But to give "three-twenty fourths" a fractional meaning, they needed to compare the part to the whole.

Constituting a connected number (a 3/24-stick) *as a fractional number* as well as a number which denotes both numerosity and length, without making an explicit part-to-whole comparison, apparently requires the construction of the splitting operation. In that case, the child would be aware of the connected number (the 3/24-stick) as a composite unit item containing three equal units which could be produced by iterating any one of the three unit items it contains (a whole-to-part relation). That is, the child would be explicitly aware of the multiplicative relation between the connected number as a composite unit item and any one of its parts, in that, say, three-twenty fourths is three times one-twenty fourth and in that one-twenty fourth iterated three times is three-twenty fourths. This opens the possibility of three-twenty fourths being considered as a fractional number because its fractional meaning would no longer be dependent on its relation to a whole of which it is a part. Rather, it would take its fractional meaning from the part of which it is a multiple. The relation to the whole of which it is a potential part would be inferential in that it could be established by means of reasoning of the sort "this stick is three-twenty fourths of the whole stick because it is three times one-twenty fourth of the whole stick." Neither Jason nor Laura independently generated this sort of a reason for why they called the 3/24-stick "three-twenty fourths" or the 6/24-stick "six-twenty fourths." Had they said something equivalent to "this stick is six-twenty fourths because it is twice three-twenty fourths," this would have been an indication of a multiplicative relation.

A further constraint in the production of fractional language

So, children's production of even the most simple of fractional language is both enabled and constrained by their current operations. In the discussion of Protocol V, my claim was that the children's lack of the equi-partitioning scheme for connected numbers constrained the production of more complex fractional language involving the production of a unit fraction commensurate with a given fraction. As that protocol occurred in the teaching episode held on the 2nd of December and the current teaching episode was held on the 7th of March of the same school year, it is informative to investigate if the children had made any progress in conceptualizing a unit fraction commensurate with six-twenty fourths.

Protocol IX: (Cont.)

T: Can you find another name for that piece (the six-twenty fourths)?

J: Ah, let's see … (Makes three copies of the 6/24-stick and aligns them end-to-end with the original 6/24-stick directly beneath the 24-stick.) Ah, four-twenty fourths.

T: It was six-twenty fourths. And now it's four … .

J: Ah, see, there's ah (moves the last 6/24-stick back and forth) … ah, if it's, see, that can do it … .

L: Six times four is twenty-four.

T: Six-twenty fourths (apparently, he did not hear what Laura said).

L: No.

T: (To Jason) what did you think of?

J: That's six-twenty fourths.

T: (Asks Jason to erase the marks from one of the 6/24-sticks. He then points to that stick and to the 24/24-stick.) How much is this stick of the whole stick?

J: Four-twenty fourths. *L:* One-twenty fourth. *T:* One-twenty fourth?

J: One-twenty fourth, as you put three more there is four-twenty fourths.
T: So as you change the marks makes it four-twenty fourths or one-twenty fourth. Ok? One more thing: What would be the share of twelve people?
L: Twelve people?
T: From that cake (pointing to the 24-stick).
J: (Copies a stick from the ruler.)
T: Oh, you are bringing in another cake.
J: That 12?
T: That cake, twelve people, how much would they get.
J: (Pulls a part out of the 24-stick and repeats it twelve times.) twelve twenty-fourths!

I consider Jason's answer of four twenty-fourths and Laura's answer of one-twenty fourths as contraindication that they used the unit Jason established containing four composite units of six items each in further operating. The way in which Jason made four 6/24-sticks indicates that the used his equi-portioning scheme in producing the four sticks. Laura's comment, "Six times four is twenty-four," is how I interpret Jason's meaning for what he produced, so I turn now to a further analysis of what I believe that the phrase didn't mean for both children.

I hypothesize establishing one of the 6/24-sticks as one-fourth of the 24-stick involves establishing a multiplicative relation between the composite unit, twenty-four, and the composite unit, six, in the sense of the multiplicative relation between an unmarked stick and one of its hypothetical parts in the splitting operation. In this case, the child would need to begin with the intention of finding a composite unit such that the composite unit twenty-four is four times that composite unit. This implies positing a hypothetical composite unit, such that, if iterated four times, would produce the composite unit, twenty-four.[9] Given that neither child had constructed the splitting operation, I consider positing such a hypothetical composite unit far removed from their current operations.

The situation of an equi-portioning scheme involves sharing, say, twenty-four items among, say, four people. The child generates a sense of possibility, and the source of this possibility originates from positing two composite units, four and twenty-four, and disembedding some composite unit from the latter and distributing it across the elements of the former. When the child disembeds a composite unit and finds that it works, this closes the scheme's activity. There is no feedback from the result, a unit of four composite units of six, into the situation in such a way that the situation is restructured in terms of the result. The construction of such a feedback system requires a review of the operations of disembedding and distribution in such a way that the operations are interiorized. In that case, the operations would be available to conscious thought and the child could then intentionally restructure the situation into a unit containing four composite units each containing six items.

What this would mean is that the child, in a situation of equi-portioning, could structure the situation prior to engaging in the activity of the scheme. That is, the child could establish an image of a unit containing a sequence of composite units, disembed one of these composite units, and anticipate iterating it a sufficient number of times to produce the numerosity, twenty-four. In this case, I regard the composite unit as an iterable composite unit and the scheme as an *equi-partitioning scheme for composite units*. Note that such an equi-partitioning scheme does not involve distributing the partitioning across each element of the composite unit being partitioned as does the equi-partitioning scheme

9 Jason and Laura could operate in some situations as if they could posit such a composite unit by using their equi-portioning scheme.

for connected numbers. Whether it would have been sufficient for Jason (or Laura) to have constructed the operations of the *equi-partitioning scheme for composite units* to say that six-twenty fourths is also one-fourth is problematic because to independently and conceptually produce this equivalence implies that they can operate on the structure of the composite unit they anticipate producing. In other words, the children may need to construct a splitting operation for composite units that is parallel to the splitting operation for the connected number, one, in order to produce the composite unit fraction as a result of productive thinking. That is, they may need to establish a multiplicative relation between a composite unit and one of its hypothetical parts.

An attempt to occasion the children's construction of improper fractions

To have constructed splitting operations, it is not sufficient for the child to start with a stick and then produce another stick such that the one given is a fractional part of the other. The reason it is not sufficient in this case to infer splitting operations is that producing the other stick would emphasize the child reversing the direction of the operations used to produce the fractional part rather than an immediate apprehension of the involved structural relations.[10] That is, to claim that a child has constructed the splitting operation, one must be able to infer that the child is aware of an unmarked stick as a unit whole and, at the same time, that the child is aware that the unit whole is partitioned into, say, eight parts. More importantly is the ability to make the inference that the child is aware that the unit whole is eight times longer than any of its parts and that any part is one-eighth of the unit whole.

One way to infer splitting operations is to observe the child independently produce improper fractions or to observe the child posing situations to another child that would involve producing an improper fraction. On the 31st of March, the teacher engaged the children in posing "I am thinking of a stick" situations. For example, Laura drew a stick and said, "I am thinking of three-elevenths of this stick." and Jason was to make the stick Laura was thinking of. Laura did eventually pose the task of "I am thinking of a stick that is eleven-elevenths of that stick!" But neither child posed a task involving an improper fraction, so the teacher intervened and posed the task of Protocol X: "Now, I am thinking of a stick that is twice as long as this six-elevenths."

Protocol X: The creative production of improper fractional language.

T: Now, I am thinking of a stick that is twice as long as this six-elevenths. Let Laura do it, you did it last time.

L: (Makes a copy of the 6/11-stick and then repeats the copy once using REPEAT, making a 12/11-stick. She then drags this stick to the end of the 6/11-stick.)

T: Now, I have a question. (Points to the 11/11-stick.) Is that the original one we started with? (Both children indicate "yes.") How much is this one of the … .

J: (Interrupting the teacher.) Twice as long as the green one (the 6/11-stick was green).

T: Original one?

J: Oh! How much is it? It is … .

L: There is only one left over from this one (the original stick).

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10 Both Jason and Laura learned, given a 7/10-stick, to partition the stick into seven equal parts and to use the seven parts to produce the 10/10-stick on the 3rd of May.

J: There is eleven, there are twelve pieces and people come to the party and they take eleven, so there is one more on.
T: So, how much is it?
J: So it is eleven, twelve-elevenths!!
T: Twelve-elevenths. (To Laura) what do you think?
L: I don't know.
T: How did you figure it out (to Jason)?
J: (Pointing to the 12/11-stick) There's six, and six plus six is twelve, and there's eleven here (pointing to the 11/11-stick).
T: What do you say?
L: Yes.
T: (To Jason.) You know what? Make a stick three times as long as this six-elevenths (points to the 6/11-stick) And you (speaking to Laura) will tell me how much it is.
J: (Makes a copy of the 6/11-stick and uses repeat to make an 18/11-stick.)
T: So, it is three times as long as the six-elevenths. So, Laura, how much is of that one?
L: Bah, bah, bah — eighteen-elevenths!!
T: How did you know that?
L: Three times six is eighteen.
T: I see! Now you used what Jason explained to you before, to do the same thing with the three?
L: Yes.
T: All right! What if I would ask five times as long as the six-elevenths?
J: Thirty.
L: (After Jason talked) Yeah, thirty-elevenths.

It was never a goal of the children to produce a fraction greater than the whole before the teacher intervened. So the teacher decided, given that the children were working at the upper boundary their fractional schemes made possible, to introduce a new but yet available operation into their schemes — that of *iterating* a composite unit. Although it was problematic whether the children's composite units were *iterable*[11] units, they could engage in iterating a composite unit a specific number of times, an operation which was made possible by their progressive integration operations.[12]

After the teacher seized upon the moment and asked Laura to make a stick that was twice as long as the 6/11-stick, Laura actually produced the stick by making copy of the 6/11-stick and then repeating the copy once using REPEAT, making a 12/11-stick. Laura immediately answered that there was only one left over after the teacher asked the children "How much is it?" which indicates she compared the original 11/11-stick with the 12/11-stick. But, her saying that she did not know if it was twelve-elevenths indicates that she was aware of a 12-part stick, but each part had lost their status as 1/11 of the original stick. Her choice to say "eighteen-elevenths" after Jason had made a stick three times as long as the 6/11-stick rather than simply "eighteen" apparently was because both the teacher and Jason had used "elevenths" in referring to the 12/11-stick. In fact, she hesitated before saying "elevenths" when she said "eighteen-elevenths." Nevertheless, she seemed to have learned a way of

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11 The distinction between iterating a composite unit and judging a composite unit as iterable is easy to make. A child might be able to engage in iterating three five times to produce fifteen, but be quite unable, given the fifteen items produced by iterating three five times, find how many units of three items each are placed with the fifteen items if there are twenty seven items total after the placement (cf. Steffe, 1992).

12 Progressive integration operations are indicated by a child counting, five and five more are ten, and five more are fifteen. The child unites the two composite units ten and five together, and then drops down to the level of the elements they contain and unites these elements together.

acting and speaking that was contingent on the teacher directives to iterate a fractional part of a stick and her ability to do so (she said that "three times six is eighteen" in justifying why she said "eighteen-elevenths") was based on her ability of iterating a composite unit.

Jason's comment, "So there are twelve pieces and people come to the party and they take eleven, so there is one more" also indicates a comparison between the original stick and the 12/11-stick Laura made. In that he chose to speak in terms of people coming to a party, the claim that his way of thinking in the context of TIMA (sticks), was also his way of thinking about his more or less everyday situations is corroborated. However, his comment indicates that he thought of only 11 people each taking a piece of the original, and so one piece would be left over, which is quite similar to the way Laura seemed to think about the situation. So, there seemed to be a lacuna in his reasoning in that he did not regard each of the 12 people as having a part of the original — only 11.

Nevertheless, his comment "twelve-elevenths" does constitutes an independent and creative production of fractional language that was based on the operations he used to produce it — "There's six pieces and six pieces so there's twelve, and there's eleven here." What seemed to be lacking is a reversal of the direction of his part-to-whole comparisons. He could disembed a part from a whole which itself consisted of parts (e.g., six parts from eleven parts) without destroying the whole. He could also inject the part into the whole both before and after he actually disembedded it from the whole. "Six-elevenths" meant "six parts out of eleven equal parts" and indicated how much the six parts were of the eleven parts. For the meaning of "twelve-elevenths" to transcend this part-to-whole meaning, he would need to inject the whole (the 11/11-stick) into the 12/11-stick with the potential to disembed the whole from what was formerly only considered as a part (in other situations like six-twelfths). Thus, he would need to restructure what was formerly a part (12/11-stick) into a composite unit containing the original whole unit (the 11/11-stick) and another unit (the 1/11-stick). This involves what I have called splitting operations because the child has to take the 11/11-stick as a unit containing hypothetical parts each of which can be iterated 11 times to produce the whole. In this way of thinking, a unit fraction (a hypothetical unit part of the 11/11-stick) becomes a fractional number *freed from its containing whole* and available for use in the construction of a 12/11-stick. Upon the emergence of the splitting operation, I regard the partitive fractional scheme as an *iterative fractional scheme*.

The hypothesis that the two children had in fact constructed the iterative fractional scheme was tested on the spot.

Protocol XI: Failure to structure fourteen-eighths as eight-eighths and six-eighths.

T: I am thinking of a stick that is fourteen-eighths of that stick (points to an 8/8-stick).

J: (Erases all marks on the 8/8-stick, partitions the resulting blank stick into fourteen parts using PARTS) fourteen-eighths?

T: Fourteen-eighths.

J: (Fills the first six parts of the 14/14-stick he made using FILL. He then activates PULL PARTS and clicks on seven of the remaining eight parts from the right to the left. As he attempts to click on the eighth part from the right endpoint of the stick, PULL PARTS deactivates. So, he pulls the seven parts he clicked on. He then reactivates PULL PARTS and clicks on the first stick from the right and pulls it out and joins it to the 7/14-stick he made, making it into an 8/14-stick.)

T: (To Laura) how much is it?

L: Eight-fourteenths.

T: I asked about fourteen-eighths and you said this is eight-fourteenths. Why?

L: Because there is fourteen little marks and eight in them.

T: That's eight-fourteenths, she said. (Indicates to Jason that he is to make fourteen-eighths and challenges the children to make it. However, both sit quietly, so the teacher makes a copy of the 14/14-stick without marks upon suggestion by a witness) can you think of another way to make fourteen-eighths? (Asks each child to explain a way to make fourteen-eighths to the other child.)

L: (Points to the blank stick) but that one is eight out of fourteen.

T: (Asks the children if that would make fourteen-eighths, attempting to be non-evaluative.)

J & L: (Sit quietly.)

T: I am thinking of a stick that is seven-eighths of this one (the blank stick).

J: (Partitions the stick into eight parts, colors the first seven, activates PULL PARTS and pulls the first filled part from the stick and makes a 7/8-stick using REPEAT.)

T: Now, I am thinking of stick that is twice as long as this stick (pointing to the 7/8-stick).

L: (After several attempts, repeats a 7/8-stick she made into a 14/8-stick.)

T: Is this twice as long Jason?

J: Yes it is.

T: How much is it?

J: No it is not (mutters. He says that Laura should have used sevenths and not eighths).

T: (The teacher clears and asks again if any of them know how much the 14/8-stick is of the original one.)

J & L: I don't know.

Jason used his partitive fractional scheme in an attempt to make fourteen-eighths, and as a consequence, he made eight-fourteenths instead. The tactic of making a stick twice as long as the 6/11-stick to make a 12/11-stick in Protocol X led the children to produce improper fractional language. But, their creative production of language was enabled by their knowing that twelve is one more than eleven and their units-coordinating scheme for whole numbers. There is no doubt that Jason, and perhaps Laura as well, established a comparison between twelve and eleven. But that was only after they iterated the 6/11-stick to produce the 12/11-stick, and this iteration was provoked by a directive of the teacher rather than being independently produced by the children. Had the children reenacted the operations they used to make a 12/11-stick to make a 14/8-stick in the original problem of Protocol XI, this would have been solid indication that they had made an accommodation in their partitive fractional scheme in the production of improper fractional language. In that Jason resorted to making an 8/14-stick, there is no indication of such an accommodation.

It is especially revealing that after the teacher returned to the situation he had used to bring forth the production of improper fractional language into the children's partitive fractional scheme in Protocol XI, that neither Jason nor Laura could say that the stick that was twice the 7/8-stick was a 14/8-stick. So, the advancement they seemed to make in Protocol X was restricted to the situation and how it was composed. It was a temporary advancement that appeared based on the teacher's directives and their use of their knowledge inherited from their number sequences.

Discussion

Connected numbers

In the formulation of the reorganization hypothesis (Steffe & Olive, 1990), I did not assume that children use their number sequences in the production of continuous units. Rather, the assumption was that children have constructed continuous units alongside of their construction of the discrete units of their number sequences. This assumption finds credibility in the analysis of the development of the unit of length given by Piaget, Inhelder, & Szeminska (1960).

> Unlike the unit of number, that of length is not the beginning stage but the final stage in the achievement of operational thinking. This is because the notion of a metric unit involves an arbitrary disintegration of a continuous whole. Hence, although the operations of measurement exactly parallel those involved in the child's construction of number, the elaboration of the former is far slower and unit iteration is, as it were, the coping (capping) stone to its construction. (p. 149)

It is important to understand that a unit of length for Piaget et al. (1960) was an iterable unit. In fact, they found that operational conservation of length, which "entails the complete coordination of operations of subdivision and order or change of position" (Piaget et al., 1960, p. 114), to be attained by three-quarters of the children they studied from 7.5 years to 8.5 years.

In that both Jason and Laura had constructed the explicitly nested number sequence and were 9 years of age, I did assume that the operations of subdivision and iteration were available to them in the context of continuous units. I did not set out with the assumption that we needed to induce these operations in them through their use of their number sequences to make fair shares. Rather, I assumed that the operations that Piaget et al. (1960) called subdivision and iteration would emerge in the context of making fair shares of sticks in TIMA (sticks), and that they would be available to the children as they learned to use their numerical concepts as templates for partitioning unmarked sticks into so many equal parts. This assumption was justified by Jason and Patricia cutting a stick into two equal parts using visual estimation in Fig. 1 and by Jason's construction of the equi-partitioning scheme in Protocol II. Jason's independent iteration of the part he broke off three times in an attempt to find if the part was a fair share had its origin in the operations he constructed in the case of continuous quantity, and it would be a misconstrual of the reorganization hypothesis to claim that his iterating the part to find if it reconstituted the whole had its origin in his numerical concept, four.

The assumption that I made concerning operative subdivision and iteration also finds justification in Laura's attempt to segment a stick into three equal parts in Protocol III. In fact, I found Laura's behavior in making the segments an occasion to at least partially explain her concept of length and length units. Her numerical template, three, served as a guide in her segmenting activity, but it did not supply the conceptual operations she used in segmenting. In both of the cases of Jason and Laura, my claim was that the children's numerical concepts were modified in the production of connected numbers. As indicated by Protocols VI and VII in which both Laura and Jason used their concepts of ten and eight as templates for making estimates of one-tenth of a stick and making the share for one of eight people, connected numbers are produced by the operations of partitioning an unmarked stick into so many equal parts and in iterating one of the parts to justify if the estimate or share is fair. In fact, I regard a connected number as produced as a result of children using their equi-partitioning schemes.

Pointing to the similarity in the construction of quantitative operations in the continuous and discrete cases should not be taken to indicate that children's construction of connected numbers emerges spontaneously. But, it does open the possibility that the construction can be provoked in children's mathematical education. The role of the teacher is to bring forth the operations that produce connected numbers, and, once they are brought forth, to use them in bringing forth children's construction of the partitive fractional scheme. Like the equi-partitioning scheme on which it is based, the partitive fractional scheme does not emerge as a product of spontaneous development. Rather, this scheme is occasioned by mathematics teachers in children's mathematical education. The scheme is certainly based on the products of spontaneous development, but I regard it as a concept in the mathematics for children[13] who have constructed the explicitly nested number sequence.

Fractional connected number sequences

The segment of the case study of Jason and Laura that I presented as Protocol IV does suggest that mathematics teachers can bring forth children's construction of connected number sequences by means of generalizing assimilation. In this way of bringing forth connected numbers, the operations involved do not include partitioning operations or what Piaget et al. (1960) called "subdivision." Rather, by treating a continuous unit item as if it were a discrete unit item, both Laura and Jason established connected numbers by making sticks so many times longer than a unit stick. In this construction, they used their operations of iterating a connected number, one, and uniting the results of the iterations into connected numbers of length and numerosity greater than one. In fact, based on their construction of the successor of a connected number using the operation of making the next number by adding one more, the children seemed to have constructed a connected number sequence. Tania and Rebecca, two children of the teaching experiment who had constructed only the tacitly nested number sequence, also constructed connected number sequences in ways that were not unlike the ways that Jason and Laura used to construct their connected number sequences (Steffe & Wiegel, 1994).

The significance of the connected number sequences for the connected number, one, other than generalizing their numerical counting schemes, emerges through the children's use of the equi-partitioning scheme to establish connected numbers that are not yet a part of a sequence. This opens the possibility for children's connected number sequences to emerge as schemes for measuring length in that case where the connected numbers established by means of partitioning are used in the production of a sequence of connected numbers of a particular kind. Measuring length involves fractional parts of unit lengths, and it requires children to have constructed what I called the iterative fractional scheme. In that case where there is only one fractional unit involved, the child's iterative fractional scheme can be used to produce a *fractional connected number sequence*. A fractional connected number sequence is a number sequence that can be used to imagine partitioning a connected number, one, into any specific number of equal parts and, further, to imagine using any one of these unit fractional parts to produce a particular connected number of either more or fewer parts than the number of parts of the partition. In the case where the iteration can be considered as continuing on indefinitely, I would say that the child has

[13] I regard mathematics for children as mathematics of other children who can be judged to have constructed similar quantitative operations. Mathematics of children consists of my explanations of children's mathematics that I am able to bring forth in children.

produced a fractional connected number sequence. In this case, the child can produce a connected number sequence for each unit fraction they can make. This opens up a powerful constructive itinerary that is totally unrealized in the teaching of fractions. Neither Jason nor Laura constructed the iterative fractional scheme while they were in their fourth grade in school, so the possibility of these two children constructing a fractional connected number sequence went unrealized. But both children did construct connected number sequences using the connected number, one, by means of generalizing assimilation of their explicitly nested number sequence.

The constructive path for producing fractional connected number sequences is much more demanding than generalizing assimilation. It involves constructing the splitting operation, and there appears to be a discontinuity between the equi-partitioning scheme and splitting operations. In the former, partitioning and iterating are operations that are more or less sequentially performed, whereas in the splitting operation, the child's awareness of a multiplicative relation between a whole and one of its hypothetical parts is produced by the composition of partitioning and iterating. In other words, they are realized simultaneously.

My current hypothesis is that the composition is produced by interiorizing the operations of partitioning and iterating. Such an interiorization produces vertical learning, and I think of it as a metamorphosis of the scheme undergoing interiorization. A metamorphic accommodation is much like the strong form of Piaget's (1980) reflective abstraction:

> Logical–mathematical abstraction ... will be called "reflective" because it proceeds from the subject's actions and operations ... we have two interdependent but distinct processes: that of projection onto a higher plane of what is taken from the lower level, hence a "reflecting," and that of "reflection" as a reorganization on the new plane. (p. 27)

When the projecting and reorganizing operations are already available at the level at which a scheme is reorganized, it is sometimes difficult to distinguish between a generalizing assimilation and a reflective abstraction. However, when these operations are not available at the level at which a scheme is reorganized, they must be assembled in experiential situations, and the projection from one level to the next may be a protracted process. In these cases, I consider the operation of projection to be set in motion by the interiorization of actions or operations carried out at the experiential level by means of reprocessing completed actions or operations in the service of a local goal.[14] As neither Jason nor Laura constructed the iterative fractional scheme, it was not possible to engage in a retrospective analysis of their case study in search for what might have appeared to be temporary modifications in their ways of operating that preceded the reorganization. A good candidate is their apparent production of improper fractional language in Protocol X. I explained it as a temporary modification of their partitive fractional scheme, but whether it engendered the iterative fractional scheme remains unknown.

Two distinct learning levels

The distinction that I made between partitive fractions as parts of fractional wholes produced as the result of using the partitive fractional scheme, and fractional numbers as multiples of a hypothetical part of a fractional whole has its justification in the concept of the splitting operation as a reorganization of the equi-partitioning scheme. Upon the emergence of the splitting operation, I consider children to be at a learning level above the learning level that is made possible by the equi-partitioning scheme. I documented in

14 See Steffe (1994) for a model of the interiorization of acts of counting that produce the initial number sequence.

Protocol V that neither Jason nor Laura could use their multiplying schemes in the establishment of composite unit fractions, such as one-eighth as consisting of three-twenty fourths of a stick. I interpreted their answer of three-eighths as a necessary error and as contraindication that the children had constructed an equi-partitioning scheme for connected numbers greater than one.

Our attempt to bring forth children's multiplying and dividing schemes to construct composite unit fractions was done to explore whether these schemes could be used as a basis for alleviating shortcomings other researchers have found in children's concepts of unit fractions. For example, Nik Pa (1987), in interviewing nine 10- and 11-year-old children, found that they could not find 1/5 of ten items because "one-fifth" referred to one in five single items. The children separated a collection of ten items into two collections of five and then designated one item in a collection of five as "one-fifth." Nik Pa's finding was quite similar to what "sixths" meant for a 9-year-old child named Alan who thought "sixths" meant "six in each pile" (Hunting, 1983). Hunting found Alan's case to be representative, in its broad outlines, of those of the other 9-year-old children he studied. Moreover, in his famous study on the grade placement of arithmetical topics, Washburne (1930) reported that a mental age level of 11 years 7 months should be attained by children if at least three out of four of them were to score 80% on a test assessing their meaning of "grouping" fractions."[15] (p. 669)

In view of the results of these three studies, an appreciation can be gained for why I considered Jason's and Laura's errors as what one might expect from children of the same age. But rather than regard the errors as simply age related, and hence, as maturational, I advanced an explanation of the operations that I see as necessary to eliminate the error. These operations involve partitioning a connected number greater than 1 using a composite unit as a template for partitioning. The way I explained these operations finds justification in a study by Lamon (1996). Two of the tasks she presented to 123 children in the fourth and fifth grades were to share four pepperoni pizzas among three children, and four oatmeal cookies among six children. Lamon (1996) reported the following percentage of children who displayed incomplete, incomprehensible, or invalid strategies: approximately 50% of the children in the case of the pepperoni pizzas; and approximately 57% of the children in the case of the oatmeal cookies. These percentages are sobering when considering that the children in the sample were at least 9 years of age, an age where I would expect over 75% of the children would have constructed the explicitly nested number sequence. They indicate the lack of construction of the equi-partitioning scheme for connected numbers, because the way I envision children using it is precisely the way in which Lamon reported the successful children operating. My current hypothesis is that the splitting operation is the kernel operation that makes possible the construction of the equi-partitioning scheme for connected numbers as well as the iterative fractional scheme. I have already explained the role that I see the equi-partitioning scheme playing in children's construction of fractional numbers. So, I regard the equi-partitioning scheme for connected numbers, the iterative fractional scheme, and fractional numbers as at a learning level above Jason's and Laura's learning level.

[15] A "grouping" fraction involved a composite part of a unit. For example, when showing children a picture of three piles of five pennies each, they were asked what part of the pennies were in each pile. In another example, when showing a picture of five piles of three pennies each with a ring around two piles, the children were asked what part of the pennies had a ring drawn around them.

Final comments

I have opened many more questions concerning the reorganization hypothesis than I have provided answers for. But rather than consider these questions as disconfirmations of the reorganization hypothesis, I consider them as a mark of a progressive research program (Lakatos, 1970). I also consider the emergence of the children's partitive fractional schemes as a major contribution to understanding children's fractional schemes, because this is the first scheme that I would be willing to call a fractional scheme. Regarding a fraction as being produced as a result of using a scheme of a certain kind is quite different than considering fractions in a context where a unit is subdivided into a whole number of parts and a certain number of these parts are then distinguished by some means (e.g., Long & DeTemple, 1995, pp. 371–372). The difference is basic in that in the former approach, fractions are regarded as products of children's operating, whereas in the latter approach, fractions are regarded as mind independent objects whose meaning is established by interpreting "fractional models," such as the set model, the region model, fraction strips, or the number line model. These models are considered to supply meaning for the mathematical definition of a fraction as an ordered pair of integers the first of which is the numerator and the last of which is the denominator.

In contrast to the approach to fractions in school mathematics by supplying meaning to the formal definition in conventional mathematics, I look to children's ways and means of operating in an attempt to judge what they might do to convince me that they engage in operations which I would be willing to regard as fractional operations and as producing something that I would be willing to call a fraction. These operations go far beyond visually separating a circular or regular polygonal region into visually congruent parts and shading so many of these parts. If any consideration is given at all to the child's operativity in these activities, it is in carrying out the actions in making the appropriate interpretations at the sensory-motor level. In contrast, asking a child to mark a candy stick so that it marks one of several equal shares and then asking the child to justify whether the share is fair, emphasizes the child's goal and ways and means of operating to achieve that goal. There is no need to impose a more or less conventional way of interpreting fractional language on the child, because the child's meaning of fractional language can be supplied by the child's goal directed scheme of actions and operations. In other words, the meaning of a number word or a combination of words consists of whatever scheme or schemes the child uses to give meaning to the word or combination of words.

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