The Splitting Group

Anderson Norton

Jesse L. M. Wilkins

Abstract: Piagetian theory describes mathematical development as the construction and organization of mental operation within psychological structures. Research on student learning has identified the vital roles two particular operations--splitting and units coordination--play in students' development of advanced fractions knowledge. Whereas Steffe and colleagues describe these knowledge structures in terms of fractions schemes, Piaget introduced the possibility of modeling students' psychological structures with formal mathematical structures, such as algebraic groups. This paper demonstrates the utility of modeling students' development with a structure that is isomorphic to the positive rational numbers under multiplication—"the splitting group." We use a quantitative analysis of written assessments from 59 eighth grade students in order to test hypotheses related to this development. Results affirm and refine an existing hypothetical learning trajectory for students' constructions of advanced fractions schemes by demonstrating that splitting is a necessary precursor to students' constructions of three levels of units coordination. Because three levels of units coordination also plays a vital role in other mathematical domains, such as algebraic reasoning, implications from the study extend beyond fractions teaching and research.

"If, as teachers, we want to foster understanding, we will have a better chance of success once we have more reliable models of students' conceptual structures, because it is precisely those structures upon which we hope to have some effect."

(von Glasersfeld & Steffe, 1991, p. 102)

Ernst von Glasersfeld and Leslie Steffe have pioneered programs to establish a psychological foundation for epistemology and for building models of students' mathematics. Both programs have roots in Piaget's structuralism (1970/1968). von Glasersfeld and Steffe express the central tenet of structuralism in noting that students construct their mathematical realities as structured systems and that teachers need models of these systems in order to affect students' growth. Of course, structures vary from student to student and, to some degree, from context to context. However, the utility of von Glasersfeld's and Steffe's work relies upon the premise that we can identify a common cognitive core in students' progressions from less powerful conceptions to more powerful conceptions. Our study quantitatively tests hypotheses related to this theorized cognitive core and schemes that model it. We focus, in particular, on fractions schemes.

In a previous paper (Wilkins & Norton, 2011), we presented and quantitatively tested a model for "the splitting loope"—a model of a psychological structure that describes how three mental operations form a closed system akin to that of a mathematical group. Specifically, the splitting loope is composed of partitioning, iterating, and splitting, where splitting is the simultaneous composition of the other two operations (cf. Steffe, 2004). The loope shares all the mathematical properties of an algebraic group, except for associativity. We argued that the loope is non-associative because, until students can coordinate three levels of fractional units (cf. Steffe & Olive, 2010), the result of partitioning a partition or iterating an iteration will remain indeterminate. We review those findings here and then extend them with results from a new study that demonstrates how students might transform the splitting loope into a splitting group.

The purpose of this paper is to test several hypotheses that arise from teaching experiment research concerning students' constructions of advanced fractions schemes. We also address the question of how Piaget's structuralism can help us understand the transformation of students' schemes, from part-whole conceptions of fractions to rational numbers of arithmetic (cf. Steffe & Olive, 2010). Before elaborating on the specific hypotheses that we test, we need to establish the framework from which they arise. We begin by outlining Piaget's structuralist approach to explaining how mathematical knowledge develops and how it differs from other knowledge. We then summarize results from teaching experiments that contribute to Steffe's program. Finally, we share results from previous quantitative tests of hypotheses that have arisen from that program.

Mathematical Structures and Psychological Structures

Piaget described children's knowledge in terms of structured systems of action. He demonstrated that actions have their origins in innate reflexes and that they are gradually differentiated to support goal-directed activity. Piaget referred to interiorized actions (those that can be coordinated as objects at higher levels of mental action) as operations. For example, a child might learn to partition through experiences involving sharing a collection of toys among friends. Once the child can carry out such activity in imagination, we might say that the child has *internalized* partitioning as a mental action. However, that mental action is *interiorized* only when it becomes coordinated as part of a system of operations in which partitioning can be considered one among several elements[1]. Interiorization and coordination also have the effect of "extending the scope of the action" beyond the context from which it was abstracted (Piaget, 1950/1947, p 37). Piaget referred to such coordinated systems of operations as structures, and he often used the group of displacements as an example.

Piaget (1970/1968) argued that the set of displacements in space form a mathematical group because the set is closed and contains an identity element, and because the set's elements are reversible and associative: The composition of any two displacements yields another displacement (closure); the identity element is the trivial displacement, which does not move the object at all; every displacement has an inverse displacement that reverses the first displacement by returning the object to its original position; finally, the displacements are associative (and commutative) because a series of displacements can be combined in any order to yield the same result. In *The Child's Conception of Space*, Piaget and Inhelder (1967/1948) described how children construct the displacement group, beginning from reflexes, and how that group relates to children's constructions of object permanence, the concept of continuity, and space in general. Interestingly, Piaget found that children construct space and geometry in reverse order from the historical development, beginning with topological concepts, continuing with those of projective geometry, and concluding with Euclidean concepts.

Not all of the structures Piaget described are isomorphic to groups, nor are they all algebraic structures. Piaget demonstrated that psychological structures could be classified as algebraic, topological, order relational, or some combination of those three classes. He

[1] See Steffe and Olive, 2010 for a detailed account of how students construct partitioning operations.

was surprised to find that the three classes, which he identified through his work with children, corresponded to the three mother structures of mathematics, as identified by the Bourbaki (Piaget, 1970). However, he was careful to note that mathematical structures merely describe the ways students seem to coordinate their mental actions and that students need not be aware of them as mathematical structures (Beth & Piaget, 1966).

Through his studies on children's constructions of reality (including space) Piaget determined that the key characteristic distinguishing logico-mathematical operations from other mental actions is reversibility.

> An operational system excludes errors before they are made, because every operation has in inverse in the system (e.g., subtraction is the inverse of addition, *+n-n=0*), or, to put it differently, because every operation is reversible, an 'erroneous result' is simply not an element of the system (1970/1968, p. 15)

Among algebraic structures, reversibility takes the form of inversion; among order relations, reciprocity plays that role. Piaget (1970/1968) claimed that, in topology, reversibility is separation, but he rarely discussed topology, except to mention its role in the construction of geometry and measurement. His main focus was on the coordination of algebraic structures and order relations. Thus, he considered ways in which the two corresponding kinds of reversibility were coordinated within larger structures, which he called "groupings." The INRC group is a meta-structure describing that coordination.

INRC stands for *Identity*, *iNversion*[2], *Reciprocity*, and *Correlative*. Those four elements form a psychological meta-structure that is isomorphic to the Klein Four Group—the group of four elements in which each element is its own inverse. Psychologically

---

[2] Piaget sometimes uses *negation* instead of *inversion*.

speaking, the group constitutes a meta-structure because it describes how forms of reversibility from other structures combine. Recall inversion (N) is the kind of reversibility for algebraic structures, reciprocicity (R) for order relations. The correlative (C) combines those two kinds of reversibility to form a new kind of reversibility, and the identity (I) results from combining any of the other three elements with themselves. Figure 1 illustrates the composition of these four elements.

	I	**N**	**R**	**C**
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

Figure 1. Cayley table for the INRC group.

It was important for Piaget to explain how inversion and reciprocity correlated because their respective structures underlie children's construction of number (Piaget, 1965/1941). Piaget described how the algebraic structure of class inclusion and the order relation of cardinality combine to form a grouping that supports whole number concepts. He claimed that the construction of the INRC meta-structure completed the grouping by organizing it at the level of formal operations. Similarly, our study investigates how students transform the non-associative splitting loope into the (associative) splitting group and how this transformation supports mature conceptions of fractions.

Teaching Experiments and Advanced Fractions Schemes

Teaching experiments (Confrey & Lachance, 2000; Steffe & Thompson, 2000) involve intensive, longitudinal interactions between a teacher and a pair of students, using some kind of manipulative (tangible or electronic) as a medium through which students can enact their mental operations. Using that medium, the teacher poses tasks to the students in order to provoke problem-solving activity and to test hypotheses about the students' thinking and learning. By videotaping students' responses, including their verbal and physical interactions with one another and the medium, the teacher can recursively analyze those responses and use them as indicators from which to infer models of the students' ways of operating. The structures that constitute those models are called schemes.

Operations comprise the key components of schemes, which von Glasersfeld (1995) described as three-part structures: a recognition template of perceived situations that the student might assimilate into the scheme, thus activating the scheme; an organized set of operations that are activated; and an expected result from that way of operating. Mental operations that play keys roles in fractions schemes include the following: partitioning, iterating, unitizing, disembedding, distributing, splitting, and units coordinating. Most of these operations are already involved in students' whole number schemes and are later transformed into ways of operating that support fractions knowledge (cf. the reorganization hypothesis, Olive, 1999; Steffe, 2002; Steffe & Olive, 2010). However, the latter two operations play distinctive roles in students' constructions of more advanced fractions schemes (Hackenberg, 2007). We outline those schemes here.[3]

The Partitive Fraction Scheme

[3] Readers unfamiliar with fraction schemes can refer to McCloskey and Norton (2009) for simpler descriptions.

Partitioning and iterating form the basis for the *partitive unit fraction scheme* (Steffe, 2002). Through sequential application of these two operations, students can produce 1/5, say, by partitioning a given continuous whole into five equal parts and iterating any one of those parts five times to test whether the piece is indeed 1/5 of the given whole. One-fifth, then, is understood as a unit that, when iterated five times, reproduces the whole. Students can generalize this way of operating to conceive of non-unit proper fractions as iterations of a unit fraction. For example, 3/5 becomes three iterations of 1/5. This generalization of the partitive unit fraction scheme establishes the *partitive fraction scheme* (PFS).

To recognize the significance of the PFS, compare the conception of fractions it affords to the part-whole conception of fractions: In part-whole conceptions 3/5 is simply 3 parts out of 5 equal parts making up the whole; in partitive conceptions 3/5 is three iterations of 1/5, which has a size relation with the whole (Steffe, 2003). Thus, the PFS affords size comparisons between part and whole that are not available in strictly part-whole conceptions. Still, the PFS has its own limitations, which students cannot overcome without first constructing of splitting operations[4].

Reversing the Partitive Fraction Scheme

Although students with PFS can produce 3/5 from a given continuous whole, reversing that way of operating requires additional operational development (Hackenberg, 2007; Steffe, 2010). In particular, students must interiorize the key operations of their partitive unit fraction schemes—partitioning and iterating—so that those operations become like two sides of the same coin (Norton & Wilkins, in review; Wilkins & Norton, 2011). Steffe (2004) defined splitting as “a composition of iterating and partitioning” (p.

[4] Findings from one recent study (cf. Norton & Wilkins, 2010) indicates that PFS, too, relies on splitting, but we adhere to results from Steffe’s research program.

135); once students have composed partitioning and iterating into a single splitting operation, they can solve tasks like those illustrated in Figure 2.

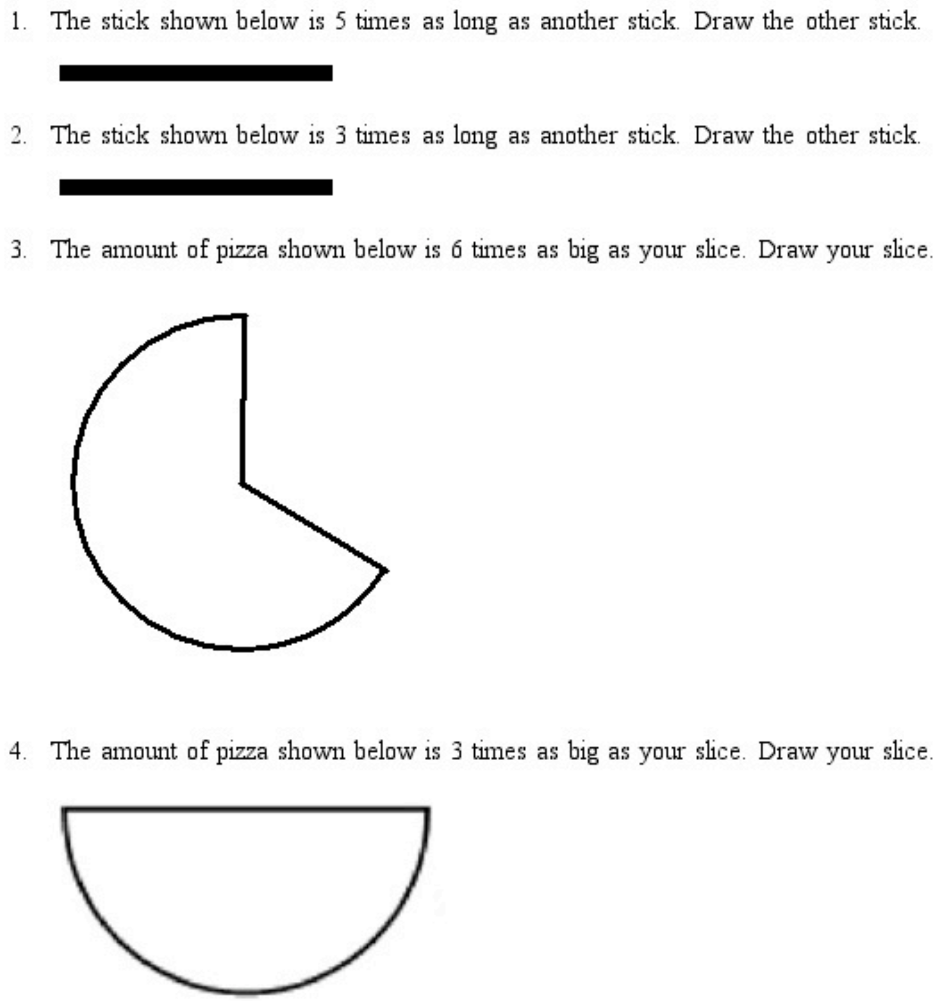


Figure 2. Tasks designed to elicit a splitting operation.

Note that successful responses require students to use partitioning to solve tasks that are iterative in nature. For example, in order to solve the first task, students must posit a piece that, when iterated five times, reproduces the whole; but to produce that piece, students must first partition. This splitting operation enables students to reverse the operations of their PFSs so that they can reproduce the whole from an unpartitioned proper fraction of it (Hackenberg, 2007). Thus, they can solve tasks like those illustrated in Figure 3.

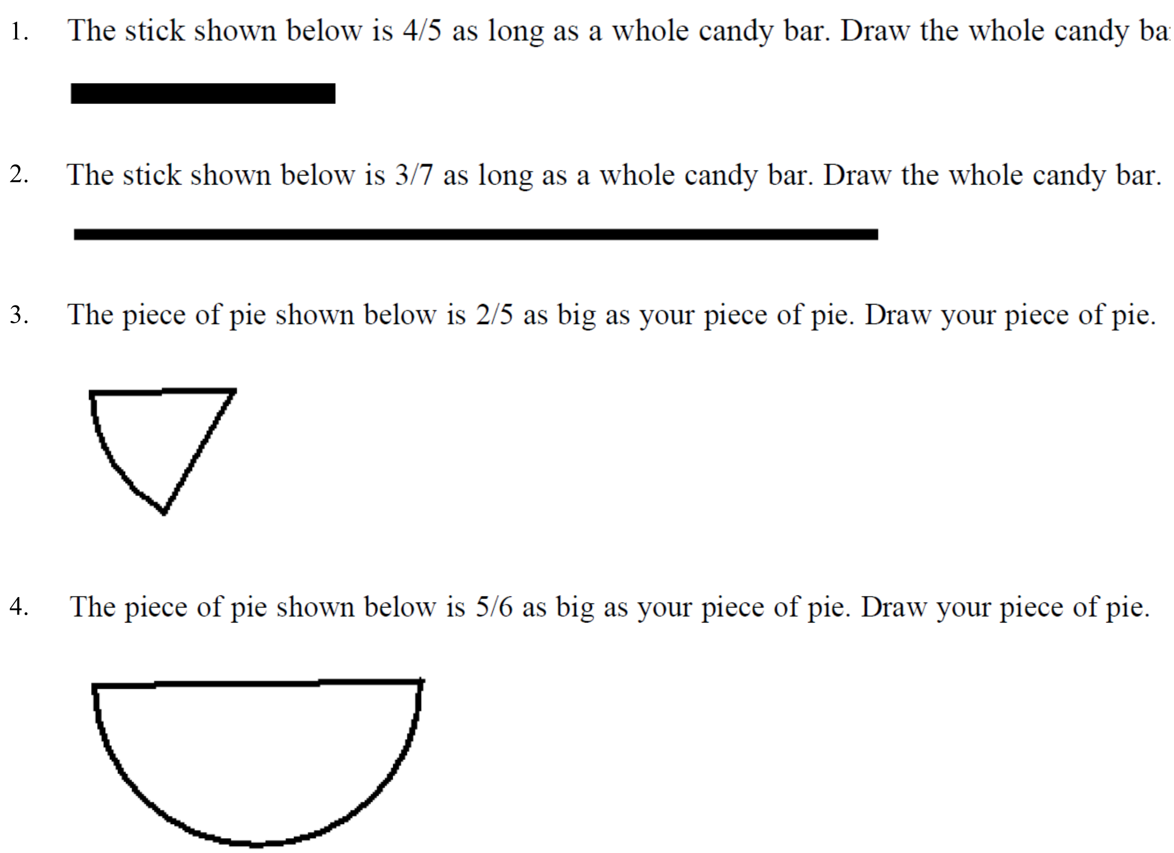


Figure 3. Tasks designed to elicit a RPFS.

As the name implies, the operations of the *reversible partitive fraction scheme* (RPFS) work in reverse from those of the PFS, and for the opposite purpose. Students use their RPFSs to reproduce the whole from a non-unit proper fraction by partitioning the fraction, m/n, into m equal parts and iterating one of those parts n times (Tzur, 2004). In addition to splitting, this way of operating involves coordinating three levels of units: 3/5 as three iterations of 1/5, five of which make the whole. However, because the three fifths are contained within the whole, students can coordinate the third level of units (the five fifths) *in activity*. On the other hand, when dealing with improper fractions, students need to be able to coordinate the three levels *prior to activity* (Hackenberg, 2007).[5]

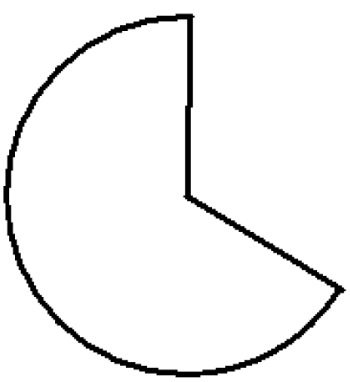
The Iterative Fraction Scheme

Students who have constructed RPFSs are still limited to working within the whole (Olive & Steffe, 2002). "For students to construct improper fractions and take them as given in further operating, bringing forth the interiorization of three levels of units is critical" (Hackenberg, 2007, p. 46). As implied by Piaget's scheme theory of learning (1950/1947), this interiorization is what makes possible the coordination of units prior to activity. Figure 4 illustrates four tasks designed to elicit students' use of three levels of units coordination.

1. The stick shown below is 2/3 of a whole stick. How many 1/9 sticks can you make from the 2/3 stick

2. The stick shown below is 3/5 of a whole stick. How many 1/20 sticks can you make from the 3/5 stick?

3. The pizza shown below is 2/3 of a whole pizza. If each person wants 1/9 of a whole pizza, how many people can share the amount shown below?



---

[5] This distinction corresponds to one Piaget (1950/1947) made in terms of anticipatory schemes. Tzur (2007) has elaborated on this distinction, referring to participatory schemes as those that a student cannot enact prior to activity.

4. The pizza shown below is 3/4 of a whole pizza. If each person wants 1/8 of a whole pizza, how many people can share the amount shown below?

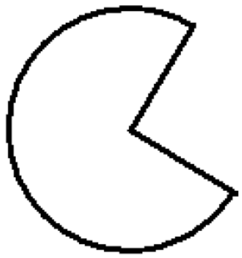


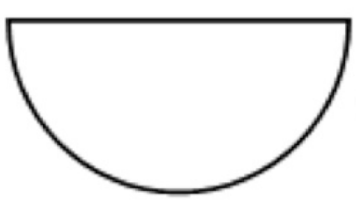
Figure 4. Tasks designed to elicit units coordination.

With the construction of splitting and the interiorization of three levels of fractional units, students are poised to construct iterative fraction schemes, which enable them to operate in the manner Hackenberg (2007) described: Students can construct, say, 7/5 as seven iterations of 1/5 and understand its multiplicative relationship with the whole so that improper fractions become "numbers in their own right" (p. 27). Students who have constructed this way of operating can solve tasks like those illustrated in Figure 5.

1. The bar shown below is 5/4 as long as a whole candy bar. Draw the whole candy bar.

2. The bar shown below is 7/3 as long as a whole candy bar. Draw the whole candy bar.

3. The piece of pie shown below is 7/5 as big as your piece of pie. Draw your piece of pie.



4. The piece of pie shown below is 5/4 as big as your piece of pie. Draw your piece of pie.

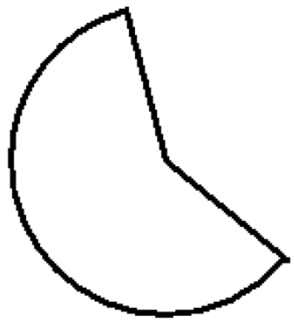


Figure 5. Tasks designed to elicit an IFS.

Theoretical Model for the Splitting Group

Teaching experiments have generated several hypotheses with strong implications for teaching, learning, and curricular design. In addition to the general progression of schemes outlined in the previous section, these include the following hypotheses about how particular schemes and operations develop:

1) "Interiorizing the operations of the equipartitioning scheme produces the composition of partitioning and iterating [splitting]" (Steffe, 2010, p. 122);
2) "Constructing a splitting operation is what allows students… to reverse the operations of their partitive fraction schemes" (Hackenberg, 2007; p. 46);
3) Whereas the reversible partitive fraction scheme depends upon splitting, generalizing this way of operating to improper fractions (i.e., the iterative fraction scheme) depends upon splitting *and* the interiorization of three levels of fractional units (Hackenberg, 2007).

Following Kilpatrick's (2001) recommendation that researchers quantitatively test such hypotheses on a larger scale, we began a project to design written assessments that could do just that (Norton & Wilkins, 2009). For example, in one previous study (Wilkins &

Norton, 2011), we designed items and built a statistical model to test the first hypothesis. Our results showed that the equipartitioning scheme, which is closely related to the partitive unit fraction scheme (Steffe, 2002), does indeed mediate the construction of splitting. A second study (Norton & Wilkins, in review) demonstrated that students who have constructed equipartitioning generally construct splitting operations soon after. The present study builds off of those previous studies but focuses on the later two hypotheses, while demonstrating how the splitting loope becomes a splitting group (see Table 1).

Table 1

Hypotheses Tested

Number	Hypothesis
2	Children construct a splitting operation before constructing a RPFS.
3.1	Children construct a splitting operation before constructing an IFS.
3.2	Children construct a RPFS before constructing an IFS.
3.3	Children interiorize three levels of units coordination before constructing an IFS.
4	Children's construction of splitting is associated with their interiorization of three levels of units coordination.
5	Children's construction of a RPFS is associated with their interiorization of three levels of units coordination.

The Splitting Loope

Like the INRC group, "the splitting loope" (Wilkins & Norton, 2011) exemplifies an isomorphism between mathematical structures and psychological structures. Mathematically, a loope is an algebraic structure with all of the properties of a group, except that loopes are not necessarily associative; in particular, the splitting loope is non-associative. Psychologically, the splitting loope describes the composition of iterating and partitioning, which constitutes splitting (Steffe, 2004). Figure 6 illustrates the mathematical structure of this composition.

	Splitting	*Iterating*	*Partitioning*
Splitting	S	I	P
Iterating	I	I	S
Partitioning	P	S	P

Figure 6. Cayley Table for the Splitting Loope

Recall that splitting enables students to solve tasks like those shown in Figure 2. We can now elaborate on the structure of the operation:

> In the splitting loope, partitioning and iterating are inverse operations. Splitting is both the identity element of the loope and the loope's fundamental aspect. To understand this, consider that splitting composes partitioning and iterating so that each operation reverses the other before any action is carried out. (Wilkins & Norton, 2011)

Note that the loope is non-associative because, until students can engage in recursive partitioning (cf. Steffe, 2003), a partition of a partition remains, simply, a partition; likewise for iterating. A student might partition a unit into three parts and then partition each of those parts into five parts, but coordinating such action as interiorized operation and anticipating the result of this way of operating (namely, fifteen 1/15 parts) constitutes a recursive partitioning operation. Hackenberg and Tillema (2009) have determined that such recursive processes require students to coordinate three levels of fractional units. In the next section, we outline a theory of how the construction of three levels of units coordination might transform the splitting loope into a splitting group.

The Splitting Group

Children begin to develop units coordination during their construction of whole number schemes, and the levels of coordination distinguish Explicitly Nested Number Sequences (ENS) from Generalized Number Sequences (GNS) (Steffe, 1992). The latter scheme involves three levels of units coordination and supports multiplicative reasoning (Hackenberg, 2010; Steffe, 1994). For example, students operating with a GNS conceive of 12 as a unit of four units, each of which contains three units[6].

Steffe's ENS and GNS number sequences elaborate on Piaget's description of number, which arises from a synthesis of order relations and class inclusion. In particular, the levels of units coordination that distinguish ENS and GNS correspond to the ways in which numbers are nested in each sequence—nestedness that Piaget described in terms of an algebraic structure for class inclusion, or "the additive grouping for classes" (Beth & Piaget, 1966, p. 176).

The main hypothesis that we test here (Hypothesis 3 above) is that the level of nestedness that defines students' transitions from ENS to GNS (namely, three levels of units coordination) also supports their construction of iterative fraction schemes. However, we note that, although GNS and IFS both involve three levels of units, the organization of these levels differs: Whereas GNS builds a unit of units of units (e.g., 24 as a unit of six units, each

---

[6] See Olive, 2000 for a cogent summary of these whole-number schemes.

containing four units), IFS builds a unit of units comprising a whole unit (e.g., 7/5 as a unit of 7 units of 1/5, which, when iterated five times, produces the whole)[7]. Thus, our hypothesis is a special case of Steffe and Olive's reorganization hypothesis (Olive, 1999; Steffe, 2002; Steffe & Olive, 2010), stipulating that children reorganize their whole number operations to construct fractions operations.

Moreover, our hypothesis stipulates that the organization of three levels of fractional units supports the transformation of the splitting loope into a splitting group. If this hypothesis is valid, then the issue of non-associativity that arises in the splitting loope could be resolved by students' development of three levels of unit coordination. Figure 7 illustrates this hypothesis through the inclusion of arrows from the reversible partitive fraction scheme and three levels of units coordinating, to the iterative fraction scheme. The figure also illustrates Hypothesis 2, through the inclusion of arrows from the partitive fraction scheme and splitting, to the reversible partitive fraction scheme. The specific hypotheses associated with the three general hypotheses are presented in Table 1.

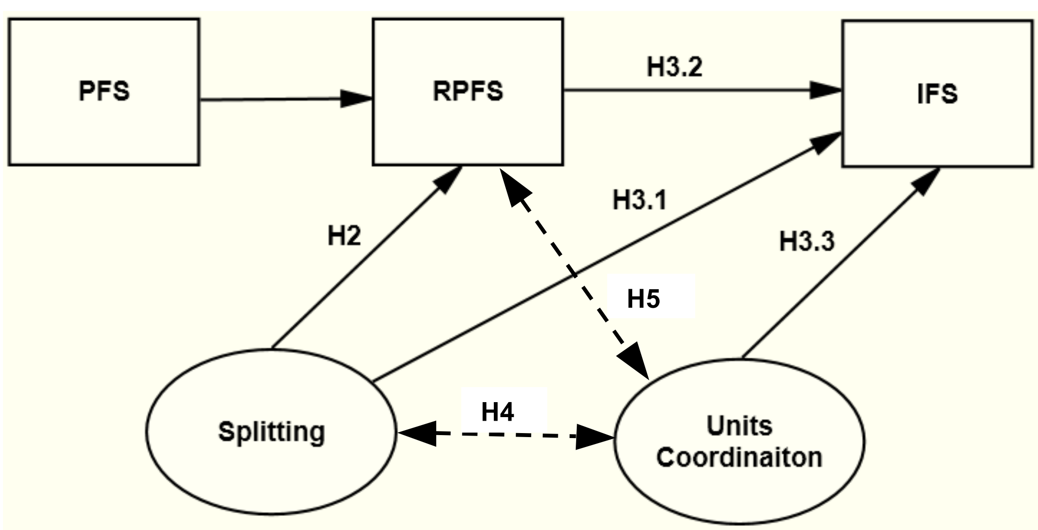


Figure 7. Theoretical hierarchy of schemes.

[7] See Hackenberg & Lee (in review) for an elaboration on this distinction.

Note: The solid arrows represent relationships hypothesized in the literature; the broken two-headed arrows represent other relationships to be investigated.

We now consider the corresponding changes in the structure of the splitting loope. Figure 6 illustrates this structure, including its non-associativity, which is evident in that some elements appear more than once in a single row or column. Specifically, partitioning composed with partitioning yields partitioning, and iterating composed with iterating yield iterating. We have argued that such idempotencies occur until students develop recursive operations, which depend upon three levels of units coordination. At that point, students should be able to resolve the compositions of partitions and iterations of various sizes. In particular, $P_n \circ P_m = P_{n \cdot m}$ and $I_n \circ I_m = I_{n \cdot m}$. When these compositions are applied to the unit segment[8], [0,1], we get examples like the following:

1) $P_5 \circ P_3[0,1] = P_{15}[0,1] = [0,\frac{1}{15}]$

2) $I_5 \circ I_3[0,1] = I_{15}[0,1] = [0,15]$

3) $I_5 \circ P_3[0,1] = [0,\frac{5}{3}]$

The first example shows that the composition of two partitions of sizes 5 and 3 yields a partition whose size is the product of 5 and 3. When this partition is applied to the unit segment, each disembedded part will measure one-fifteenth; likewise for the composition of iterations. The third example shows that when partitions are composed with iterations (or vice versa, because the operations commute), the size of the iteration multiplies the measure of the part. The group generated by compositions of I_m and P_n is isomorphic to the

[8] Like a group action on a set, in abstract algebra.

positive rationals under multiplication, $\{Q^+, \times\}$, which is precisely the mathematical structure we want students to construct.

Methods

Our model of the splitting group explains how students construct the rational numbers under multiplication through the construction of two key operations: splitting and the coordination of three levels of fractional units. This explanation aligns with Hypotheses 2 and 3, which arose from previous research. In this section, we describe our methods for quantitatively testing those hypotheses.

Participants

The study involved 59 students from five eighth-grade classrooms, all taught by the same teacher. The school is located in the rural Southeast, with 59% of its students enrolled in free-and-reduced lunch programs. Of the 59 students, one did not give written consent to have their data used in analysis. This student was removed leaving us with a working sample of 58 students.

Measures

We administered two versions of a 22-item test in May 2009. The items on the two versions of the test were identical, but the order of the items was rotated to help control for possible effects due to item order. The 22 items included four items each associated with splitting, RPFS, IFS, and units coordination. We designed these items to have differing contexts (linear or area) and, when possible, a “forward” and “backward” representation (going from a concrete representation to a numerical fraction answer, and going from numerical fraction to concrete representation). In addition, there were six items that were designed to test for the robustness of the schemes. Two of these items were not analyzed for this study.

We designed individual items to provoke responses that might indicate a particular scheme or operation. In other words, each item provided students with a situation in which to enact particular ways of operating. The two authors independently rated student responses for each item. We assessed student responses based on all of the written work associated with the item and not solely on whether they had the "correct" answer (cf. Norton, 2007). From this assessment, we inferred whether there was sufficient indication that the student had operated in a way that was consistent with a particular scheme or operation.

Following Norton and Wilkins (2009, p. 156; also see Wilkins & Norton, 2011), we scored responses to each item in the following way:

0: There was counter-indication that the student could operate in a manner compatible with the theorized scheme or operation. Counter-indication might include incorrect responses and markings that are incompatible with actions that would fit the scheme.

1: There was strong indication that the student operated in a manner compatible with the theorized scheme or operation. Indications might include correct responses, partitions, and iterations.

For item responses that represented some indication that a student operated in a manner compatible with the theorized scheme or operation, but nonetheless, did not show strong indication, we gave a score of .5. Initially, we also used scores of .4 or .6 to indicate a leaning, one way or the other. We used these scores to aid in the overall inferences about the schemes or operations.

Following procedures from our previous work (Norton & Wilkins, 2009; Wilkins & Norton, 2011), for each set of four items representing a particular scheme or operation, we independently summed the four individual item scores resulting in an overall raw score for each scheme between 0 and 4. These raw scores were then used to further infer whether a child had a particular scheme (coded as 1) or did not have it (coded as 0). If a student's overall raw score for a given scheme or operation was greater than or equal to 3, it was inferred that the student's actions were consistent with the particular scheme or operation. If a student's overall score was less than or equal to 2, it was inferred that the student's actions were not consistent with the particular scheme or operation. For students whose overall score for a scheme was between 2 and 3, the individual raters considered the student's work on all four individual items and inferred from all of the work whether there was indication that the student had operated in a way that was compatible with the particular scheme or operation. For cases in which there was disagreement the two raters re-examined the cases together to reach a consensus.

Analysis

We first conducted an analysis of inter-rater reliability to judge the level of agreement between the raters' assessments of the students' schemes and operations. We did this by calculating Cohen's a kappa statistic (Cohen, 1960) for each scheme and operation. Next we entered student data into two-by-two contingency tables in order to examine the relationships among the schemes and operations as prescribed by our hypotheses (see Table 1). We were interested in examining whether the construction of particular schemes and operations was associated with the construction of other particular schemes and operations. In this regard and with respect to Figure 7, we broke Hypothesis 3 into three parts (3.1, 3.2, and 3.3). The gamma statistic, G, is an appropriate statistic for testing the magnitude of the association between the schemes and operations, as pairs of ordered variables (Siegel & Castellan, 1988).

Furthermore, we were interested in testing whether particular schemes and operations developmentally preceded the development of other schemes and operations. For example, Hypothesis 2 predicts a direct association between the construction of splitting and RPFS, and further that the construction of a splitting operation would precede the construction of a RPFS. If the data were consistent with this hypothesis the two variables would have a positive direct association (i.e., $G > 0$) and a *weak monotonic relationship.* Visually this relationship manifests in a staircase pattern in which the data fall along the diagonal and predominantly in the lower left cell (or upper right cell; for a more detailed discussion see Wilkins & Norton, 2011). Whereas the gamma statistic would provide evidence of a relationship, it is a symmetrical measure and does not provide a test of developmental order. That is, two variables could have a strong association, but still not have a developmental relationship. A test of developmental order is done by first visually examining the off-diagonal data to see if the cell numbers are different. This is then followed by a test to see if this difference is in the hypothesized direction and statistically different than what would be expected by chance. For this we used a binomial test. For hypotheses 2, 3.1, 3.2, and 3.3 we hypothesized a direct association between the variables and also predicted a developmental order, thus we used one-tailed gamma and binomial tests. For hypotheses 4 and 5 we were interested in investigating the relationship between units coordination and splitting and units coordination and RPFS but had no a priori hypothesis about the developmental order between the variables, thus it was necessary to use two-tailed tests.

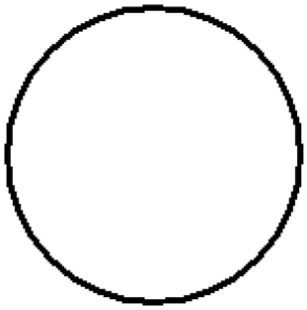
Results

Inter-Rater Reiability

Initially we selected and rated 6 of the 58 students (approximately 10%). We then compared, discussed, and reconciled our ratings for these students, and then we rated the remaining 52 students. To assess the overall agreement (inter-rater reliability) for each scheme and operation, we calculated the kappa statistic, K, for these 52 students. The kappa statistic for units coordination ($K = .90$, $p < .05$), splitting ($K = .87$, $p < .05$), and IFS ($K = .84$, $p < .05$) represented "almost perfect" agreement (Landis & Koch, 1977, p. 165). The kappa statistic for RPFS ($K = .73$, $p < .05$) represented "substantial" agreement (p. 165). These statistics indicate consistently high rater agreement across the schemes and operation. We then reconciled disagreements to create one rating per student for each scheme and operation. To assess the overall agreement (inter-rater reliability) for each of the robustness items we also calculated the kappa statistic. Respectively, the Kappa statistics for the robustness items 1-4 (Figure 8) were 0.88, 0.84, 1.00, and 0.93, all statistically different from zero ($p < .05$) and representing perfect or "almost perfect" agreement (Landis & Koch, 1977, p. 165).

1. Make a stick that is 7/5 as long as the stick drawn below.

2. Make 4/3 of the pie drawn below.



3. What fraction is the longer stick out of the shorter stick?

4. What fraction is the longer stick out of the shorter stick?

Figure 8. Robustness items

Splitting and RPFS

We present frequencies of students who had constructed splitting operations and RPFSs in Table 2. Based on Hypothesis 2, children's construction of splitting should occur prior to the construction of a RPFS. A visual inspection of the table reveals a staircase pattern—that is, the data fall predominately in the diagonal and lower left cells—providing evidence that the construction of splitting does occur prior to the construction of RPFS. Furthermore, in this case, there were no students who had constructed a RPFS who had not first constructed a splitting operation, which indicates a perfect weak monotonic relationship (G = 1.00, p < .0001, one-tailed). A binomial test (*n=27*) revealed that this finding is significantly less likely than would be expected by chance, exact binomial *p*(one-tailed) < .0001. These findings provide evidence that a splitting operation is constructed prior to the construction of a RPFS.

Table 2

Splitting and RPFS

	RPFS		
Splitting	0	1	Total
0	20	0	20
1	27	11	38
Totals	47	11	58

Note. G = 1.00, p < .0001, one-tailed; Exact Binomial p (one-tailed) < .0001.

Splitting and IFS

We present frequencies of students who had constructed splitting operations and IFSs in Table 3. Based on Hypothesis 3.1, children's construction of splitting would occur prior to the construction of an IFS. Consistent with the hypothesis, we found a perfect weak monotonic relationship between splitting and IFS (G = 1.00, p < .01, one-tailed). Considering the off-diagonal cases (n = 31), there were no students who had constructed an IFS prior to first constructing a splitting operation. A binomial test revealed that this finding is significantly less likely than would be expected by chance, exact binomial p(one-tailed) < .0001. These findings provide evidence that a splitting operation is constructed prior to the construction of an IFS.

Table 3

Splitting and IFS

	IFS		
Splitting	0	1	Total
0	20	0	20
1	31	7	38
Totals	51	7	58

Note. G = 1.00, p < .01, one-tailed; Exact Binomial p (one-tailed) < .0001.

RPFS and IFS

We present frequencies for students' construction of RPFS and IFS in Table 4. Based on Hypothesis 3.2, children construct RPFSs before constructing IFSs. While most students in the sample had not constructed either of these schemes, we did find a strong statistically significant positive association between the two schemes (G = 0.96, p < .01, one-tailed). Further, a visual inspection of the contingency table reveals a weak monotonic relationship between the two schemes suggesting that the construction of a RPFS does occur prior to the construction of an IFS. Considering the off-diagonal cases (n = 6), we find that 5 of these 6 children (83%) constructed a RPFS before constructing an IFS, exact binomial p(one-tailed) = .0512. Although this finding does not strictly reach our criterion for statistical significance (i.e., p < .05), it is reasonable to conclude that the data are consistent with our a priori hypothesis.

Table 4

RPFS and IFS

RPFS	IFS 0	1	Total
0	46	1	47
1	5	6	11
Totals	51	7	58

Note. G = 0.96*, p* < .01, one-tailed; Exact Binomial *p* (one-tailed) = .0512.

Units Coordination and IFS

We present frequencies for students' construction of units coordination and IFS in Table 5. Based on Hypothesis 3.3, in order for students to construct an IFS they must first interiorize three levels of units coordination. Consistent with this hypothesis, we found a perfect weak monotonic relationship between units coordination and the construction of an IFS (*G* = 1.00, p < .001, one-tailed). We further found that of the off-diagonal cases (*n* = 8) that none of these children had constructed an IFS without first interiorizing units coordination, which is less likely than would be expected by chance, exact binomial p(one-tailed) < .01. These findings provide evidence that children must first interiorize three levels of units coordination before they can construct an IFS.

Table 5

Units Coordination and IFS

UC	IFS 0	1	Total

0	43	0	43
1	8	7	15
Totals	51	7	58

Note. G = 1.00, *p* < .001, one-tailed; Exact Binomial *p* (one-tailed) < .01.

Units Coordination and Splitting

We present frequencies for students' units coordination and splitting in Table 6. We had no a priori directional hypothesis regarding the relationship between units coordination and splitting; however, we felt that it was useful to investigate a possible relationship in order to better understand the roles of splitting and units coordination in the construction of advanced fraction schemes. Thus, our working hypothesis was that there was a relationship between units coordination and splitting, versus the null hypothesis that there was no relationship (see Table 1).

We found a statistically significant positive relationship between units coordination and splitting ($G = 0.83$, $p < .001$, two-tailed). A post-hoc analysis of the off-diagonal cases ($n = 25$) found that only one student had constructed units coordination prior to constructing a splitting operation, which is less likely than would be expected by chance, exact binomial p (two-tailed) < .0001. These findings suggest that there is a strong relationship between children's construction of splitting and units coordination and, further, that the construction of splitting seems to occur prior to the interiorization of three levels of units coordination.

Table 6

Splitting and Units Coordination

	Units Coordination		
Splitting	0	1	Total
0	19	1	20
1	24	14	38
Totals	43	15	58

Note. $G = 0.83$, $p < .001$, two-tailed; Exact Binomial p (two-tailed) $< .0001$.

Units Coordination and RPFS

We present frequencies for students' construction of units coordination and RPFS in Table 7. We had no a priori directional hypothesis regarding the relationship between RPFS and units coordination. However, we did want to investigate the relationship in order to better understand the construction of RPFS. Thus, our working hypothesis was that there was a relationship between RPFS and units coordination, versus the null hypothesis that there was no relationship (see Table 1).

We found a statistically significant positive association between units coordination and RPFS ($G = 0.88$, $p < .01$, two-tailed). However, when examining the off-diagonal cases ($n = 10$), we found that the numbers were not different from chance, exact binomial p (two-tailed) = .206. This finding suggests that, while there is a relationship between units coordination and RPFS, interiorizing the coordination of three levels of units is not necessary for the construction of a RPFS.

Table 7

Units Coordination and RPFS

	RPFS		
UC	0	1	Total
0	40	3	43
1	7	8	15
Totals	47	11	58

Note. G = 0.88, p < .01, two-tailed; Exact Binomial p (two-tailed) = .206.

Robustness Items

Table 8 presents correlations between the four schemes/operations and the four robustness items used in our study. The first two robustness items (see Figure 8) require students to produce a specified improper fraction from a given whole: the first of these items uses a linear context; the second uses a circular area context. The last two items require students to determine the size of a given improper fractional stick relative to a given whole stick (both using linear contexts). With the exception of the second item, student performance on the robustness items correlates strongest with three levels of units coordination (UC) and the iterative fraction scheme (IFS). As a rule of thumb, in educational research, correlation coefficients around .50 and higher are considered large, while correlations around .30 are considered of medium size (Cohen, 1988). The correlations are especially strong for the third and fourth items, indicating that estimating relative sizes for improper fractions is closely associated with the construction of three levels of units coordinating and the iterative fraction scheme. On the other hand, the second item seems closely associated with splitting, perhaps because the circular area context makes it easier for students to see three thirds in the whole, and then add one more.

Table 8.

Descriptive statistics and correlations for schemes, operations, and robustness items.

	Splitting	RPFS	IFS	UC	%	*SD*
R1	.22	.27*	.31*	.34**	.22	.42
R2	.51***	.13	.08	.18	.33	.47
R3	.30*	.37**	.44***	.53***	.22	.42
R4	.28*	.40**	.59***	.48***	.21	.41
%	.66	.19	.12	.26		
SD	.48	.40	.33	.44		

Note: *$p < .05$, **$p < .01$; ***$p < .001$. *Item numbers refer to those used in Figure 8.*

Conclusions

Students can use part-whole reasoning to conceive of proper fractions as *m* parts out of *n* parts in an equally-partitioned whole, but this reasoning does not easily extend to a partitive understanding of *m/n* as *m* iterations of *1/n* (Olive & Vomvoridi, 2006; Tzur, 1999). Students in the United States generally reason with fractions as part-whole relations alone, as US textbooks emphasize the role of partitioning wholes while neglecting the role of iterating unit fractions (Watanabe, 2007). Splitting is an especially powerful operation because it composes the fundamental operations of partitioning and iterating (Steffe, 2002). In a previous study (Wilkins & Norton, 2011), we demonstrated how students construct this composition through the interiorization of their equipartitioning schemes, which arise through experiences in iterating unit fractions (Olive & Vomvoridi, 2006). However, even among students who split, partitive conceptions remain constrained by the student's ability to coordinate the various levels of units involved in fractions, especially improper fractions (Hackenberg, 2010).

The findings reported here extend previous findings by testing hypotheses about students' constructions of advanced fractions schemes—those that depend upon splitting operations. At the same time, our results further substantiate Piaget's structuralism by identifying isomorphisms between students' psychological and mathematical structures, as students progress from the construction of a splitting loope to the construction of a splitting group.

The Splitting Group

Following Piaget (1971/1968), we have modeled students' operational systems using mathematical structures, such as the splitting loope. Based on these structures, we hypothesized that students could reorganize that structure into a stronger one through the interiorization of three levels of fractional units. This new structure, which we call the splitting group, is isomorphic to the group of rational numbers under multiplication.

Our hypothesis aligns with two previous hypotheses generated through Hackenberg's (2007) teaching experiment with four sixth-grade students; hypotheses that we have broken down into Hypotheses 2, 3.1, 3.2, and 3.3 (see Table 1). These four hypotheses express directional relationships between students' constructions of splitting, reversible partitive reasoning, three levels of units coordination, and the iterative fraction scheme. Our study has affirmed all of these hypothesized directional relationships, along with a fifth, unexpected relationship. The five relationships are illustrated in Figure 9.

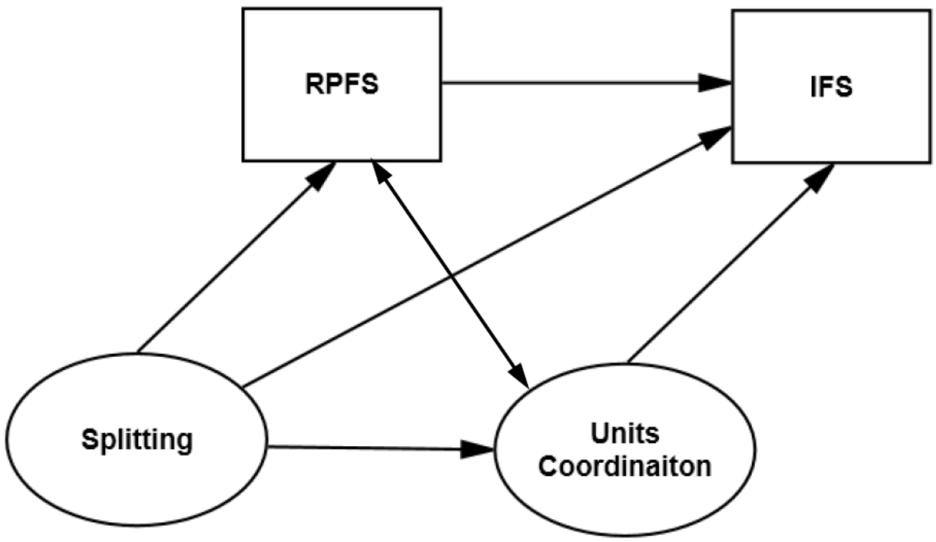


Figure 9. Relationships among the schemes and operations.

The relationships lend credence to our larger hypothesis about the construction of the splitting group. Students who can split can learn to reverse the operations of their partitive fraction schemes, but they cannot resolve, in generality, compositions of partitions and iterations of various sizes until they begin coordinating three levels of fractional units. Specifically, in addition to splitting, students need to interiorize three levels of units coordination to resolve tasks requiring them to estimate the sizes of improper fractions relative to the whole, or to reproduce a whole from a given improper fraction. The unexpected relationship between splitting and units coordinating provides further indication that the splitting loope is a structure that serves as a precursor to the construction of the splitting group, and, thus, the positive rational numbers under multiplication.

Whereas the splitting loope describes the structure of students' splitting operations, the splitting group extends that structure by resolving all possible combinations of iterations and partitions, of various sizes. We hypothesized that a general resolution would require three levels of units coordination, at least for improper fractions. Although students can often produce improper fractions, in action, from a given whole (Tzur, 1999), conceptualizing the result theoretically involves three levels of units (Hackenberg, 2007); for example, 7/5 is a unit of seven units of 1/5, five of which produce the whole unit. Likewise, reproducing the whole from an improper fraction would require navigation through those same three levels. On the other hand, reproducing the whole from a proper fraction might be resolved through the use of splitting, two levels of units, and part-whole reasoning (e.g., 5/7 is five times as big as the unit, and two more of those units are needed to make 7). Our results quantitatively affirm these theoretical distinctions by demonstrating that students can reverse proper fractions without having three levels of units coordination available, but that they need this operation in order to reverse improper fractions (see Tables 5 and 7).

Our results also indicate that the construction of three levels of units coordination enables the transformation of the splitting loope into a splitting group, especially through the construction of the iterative fraction scheme. In addition to those cited above, results reported in Table 8 support the argument by demonstrating strong correlations (.44 and .59) between student performance on two kinds of tasks: producing the whole from a given improper fraction (as assessed by IFS tasks), and estimating the size of a given improper fraction relative to a given whole (Figure 8, R3 and R4). Performance on tasks requiring students to produce an improper fraction from a given whole was not as strongly correlated with IFS, possibly for reasons described above (cf. Tzur, 1999), but in the linear context (Figure 8, R1), were correlated nonetheless. Thus, we have that, with the construction of the IFS, and on the basis of three levels of units coordination, students can resolve tasks involving the estimation, production, or reversal of fractions in general.

Three levels of units coordination is also theoretically necessary for establishing equivalence classes within the splitting group (e.g., 4/6 and 2/3 represent the same element in the group), and composing the various elements of the group (Steffe, 2003). Because the elements are fractions (proper and improper) and composition is multiplication, these compositions are products of fractions. Through his teaching experiments, Steffe (ibid) modeled students' abilities to establish fraction equivalences and to conceptualize fraction products, in terms of the *commensurate fraction scheme* and the *fraction composition schemes*, respectively. Theoretically, both of these schemes employ splitting operations and three levels of units coordination (Steffe & Olive, 2010). This theory aligns with our description of the splitting group; namely that, beyond splitting, three levels of units coordination are necessary to transform the splitting loope into a splitting group. However, our study did not afford the opportunity to assess students' commensurate or fraction composition schemes, and thus, the corresponding aspects of the group. Our new hypothesis is that, for students who can split and coordinate three levels of fractional units, the construction of these schemes—and the splitting group as a whole—is a lateral learning goal (cf. Steffe, 2004), just as it is for the construction of IFS.

The INRC Group

When students construct the splitting group, the fractions that comprise the elements of that group (all positive rational numbers) should act as operators (cf. Kieren, 1980). We have described their operation on the unit segment, [0, 1]. For example, 3/5, as the composition of a partition of size 5 and an iteration of size 3, should transform [0, 1] into the segment [0, 3/5]. Likewise, the group can transform the unit segment into a segment of any rational length. Thus, in addition to the ordering relations and class inclusions that Piaget ascribed to children's constructions of whole number, the splitting group, as a group action on a set, is governed by topological properties as well. In other words, the splitting group integrates aspects from all three mother structures: ordering relations, algebraic structures, and topological structures. The INRC group, which Piaget developed through his research on children's construction of whole number, is ill-equipped to resolve the three kinds of reversibility contained in these three structures. In particular, the correlative, "C", has nothing to do with topology, in which reversibility might be characterized as the reconciliation of "opposing notions of proximity and separation within a global, unified concept of continuity" (Piaget & Inhelder, 1967/1948, p. 149)

Alternatively, we can consider Piaget's INRC group as a set of transformations applied to the tasks themselves. Consider tasks 1 through 4, illustrated in Figure 10. Tasks 2, 3, and 4 come from the present study. The first task was posed in a previous study (Norton & Wilkins, 2010) of seventh-grade students at the same school. If we consider that task as a base case, there are two obvious ways to reverse it: reverse the order of the terms in the fraction m/n, or reverse the order of the operations in the task. Instead of asking the student to produce a proper fraction, m/n, from a given whole, we can ask the student produce n/m from the given whole (as in Task 2), or we can ask the student to produce the whole from m/n (as in Task 3). Assuming the tasks had started from the same fraction, m/n, Task 2 and Task 3 would reverse Task 1 because, when applied to the result of Task 1, they would return it to the original whole. Task 2 would be the *inverse* of Task 1 because it would use the multiplicative inverse of the fraction; Task 3 would be the *reciprocal* of Task 1 because it would reverse the direction of operating. Task 4 would be the *correlative*, because it would represent the composition of the inverse and reciprocal.

1. Your stick is 3/5 as long as the stick shown below. Draw your stick.

2. Make a stick that is 7/5 as long as the stick drawn below.

3. The stick shown below is 4/5 as long as a whole candy bar. Draw the whole candy bar.

4. The bar shown below is 5/4 as long as a whole candy bar. Draw the whole candy bar.

Figure 10. Tasks representing the INRC transformations

This description fits Piaget's (1970) application of INRC to the snail problem, wherein a snail's movements on a board can be reversed by ether reversing the direction the snail moves, or by sliding the board in the direction opposite to the snail's movement. Piaget claimed that, when students can resolve all the possible transformations of the snail/board movements, we can infer that the student has constructed INRC. At least in our case, this resolution occurs when students learn to resolve Task 4, the correlative of the base case. Task 1 is the natural choice for the base case because it only involves the production of a proper fraction from a given whole. Task 3 involves reversing that way of operating, and Task 2 involves the production of an improper fraction. Table 4 and Figure 8 (see correlation and relative percentages for R1 and IFS) indicate that students who can solve tasks like Task 4 (those used to assess the iterative fraction scheme) can solve the others as well.

A growing wealth of research on students' reasoning demonstrates that coordinating three levels of units plays a vital role in students' reasoning across several mathematical domains, including fractions (Steffe & Olive, 2010), algebra (Ellis, 2007; Hackenberg, 2010; Olive & Caglayan, 2007), and multiplicative reasoning in general (Thompson & Saldanha, 2003). These domains intersect during students' middle school years, in which students' should also be transforming their reasoning to the formal operational stage (Piaget, 1970). By tying the construction of the iterative fraction scheme to the construction of the INRC group, our study further suggests a connection between three levels of units coordination and the development of formal operational reasoning.

Implications

Results from testing hypotheses have immediate implications for underlying research on students' fractions schemes. Moreover, we have established a new directional link between splitting and units coordinating, which was not posited in previous studies. This link implies that the splitting loope serves as a necessary precursor to the splitting group, while underscoring the vital role units coordinating serves in the latter. As such, examining how to support students' constructions of splitting and units coordinating should become the major focus for future research on students' fractions schemes. For now, we can say that splitting occurs as the iteriorization of equi-partitioning (Steffe, 2002;

Norton & Wilkins, in review) and three levels of units coordinating appears to develop from splitting.

Implications from our study present a more dire challenge for middle school teachers. Whereas Common Core Standards (http://www.corestandards.org/) call for students to understand 5/4 as five iterations of 1/4, as early as fourth grade, only 12 percent of the eighth grade students in our study had constructed iterative fraction schemes, which are necessary for such a conception (see Table 8). Middle school teachers have the weight of students' academic futures on their shoulders as they attempt to shore up students' understanding of fractions, and at the same time, prepare those students for high school algebra. Research continues to coalesce around the idea that teachers can address both demands by focusing on students' interiorization of three levels of units coordination, and we now know that the splitting operation is prerequisite to this interiorization.

References

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology* (W. Mays, Trans.). New York: Gordon and Breach.

Cohen, J. (1960), A coefficient of agreement for nominal scales, *Educational and Psychological Measurement, 20*(1), 37–46.

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Lawrence Erlbaum Associates.

Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture driven research design. In A.E. Kelly & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 231-265). Mahwah: NJ. Lawrence Erlbaum.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439 – 478.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior, 26*(1), 27-47.

Hackenberg, A. J. (2010). Students' reversible multiplicative reasoning with fractions. *Cognition and Instruction, 28*(4), 383-432.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior, 28*, 1-18.

Kieren, T. E. (1980). The rational number construct—its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (125-149). Columbus: ERIC/SMEAC.

Kilpatrick, J. (2001). Where's the evidence? *Journal for Research in Mathematics Education, 32*(4), 421–427.

Landis, J. R. and Koch, G. G. (1977). The measurement of observer agreement for categorical data. *Biometrics, 33*, 159-174.

McCloskey, A., & Norton, A. (2009). Modeling students' mathematics using Steffe's advanced fractions schemes. *Mathematics Teaching in the Middle School, 15*(1), 44-56.

Norton, A. (2006). What's on your nation's report card? *Teaching Children Mathematics 13*(6), 315-319.

Norton, A. & Wilkins, J. L. M. (2009). A quantitative analysis of children's splitting operations and fraction schemes. *Journal of Mathematical Behavior*, *28*(2/3), 150-161.

Norton, A. & Wilkins, J. L. M. (in review). Supporting students' constructions of the splitting operation. *Cognition and Instruction*.

Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning, 1*(4), 279–314.

Olive, J. & Caglayan, G. (2007). Learner's difficulties with quantitative units in algebraic word problems and the teacher's interpretations of those difficulties. *International Journal of Science and Mathematics Education, 6*, 269-292.

Olive, J., & Steffe, L. P. (2002). The construction of an iterative fractional scheme: the case of Joe. *Journal of Mathematical Behavior, 20*(4), 413-437.

Olive, J. & Vomvoridi, E. (2006). Making sense of instruction on fractions when a student lacks necessary fractional schemes: The case of Tim. *Journal of Mathematical Behavior, 25*(1), 18-45.

Piaget, J. (1950). *The psychology of intelligence* (M. Berlyne, Trans.). New York: Harcourt Brace (Original work published 1947).

Piaget, J. (1965). *The child's conception of number*. New York: Norton (Original work published 1941).

Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). New York: Norton.

Piaget, J. (1970). *Structuralism* (C. Maschler, Trans.). New York: Basic Books (Original work published 1968).

Piaget, J. (1971). *Psychology and epistemology* (A. Rosin, Trans.). New York: Grossman.

Piaget, J. & Inhelder, B. (1967). *The child's conception of space* (F. J. Langdon & J. L. Lunzer, Trans.). New York: Norton (Original work published 1948).

Siegel, S. & Castellan, N. J., Jr. (1988). *Nonparametric statistics for the behavioral sciences* (2nd ed.). New York: McGraw-Hill.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. (1994). Children's multiplying schemes. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 3-39). Albany: State University of New York Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*(3), 267-307.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura (grade 5). *Journal of Mathematical Behavior, 22*, 237-295

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*(2), 129-162.

Steffe, L. P. (2010). The partitioning and fraction schemes. In Steffe, L. P. & Olive, J. (Eds.), *Children's fractional knowledge* (pp. 315-340). New York: Springer.

Steffe, L. P., & Olive, J. (Eds.) (2010). *Children's fractional knowledge*. New York: Springer.

Steffe, L. P., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Lawrence Erlbaum Associates.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *Research companion to the principles and standards for school mathematics* (pp. 95–113). Reston, VA: National Council of Teachers of Mathematics.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

Tzur, R. (2004). Teacher and students' joint production of a reversible fraction conception. *Journal of Mathematical Behavior, 23*, 93-114.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics, 66*, 273–291.

von Glasersfeld, E. (1995*). Radical constructivism: A way of knowing and learning*. London: Routledge Falmer.

von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *Journal of Educational Thought, 25*(2), 91-103.

Watanabe, T. (2007). Initial treatment of fractions in Japanese textbooks. *Focus on Learning Problems in Mathematics, 29*(2), 41-60.

Wilkins, J. L. M., & Norton, A. (2011). The splitting loope. *Journal for Research in Mathematics Education, 42*(5), 386-416.